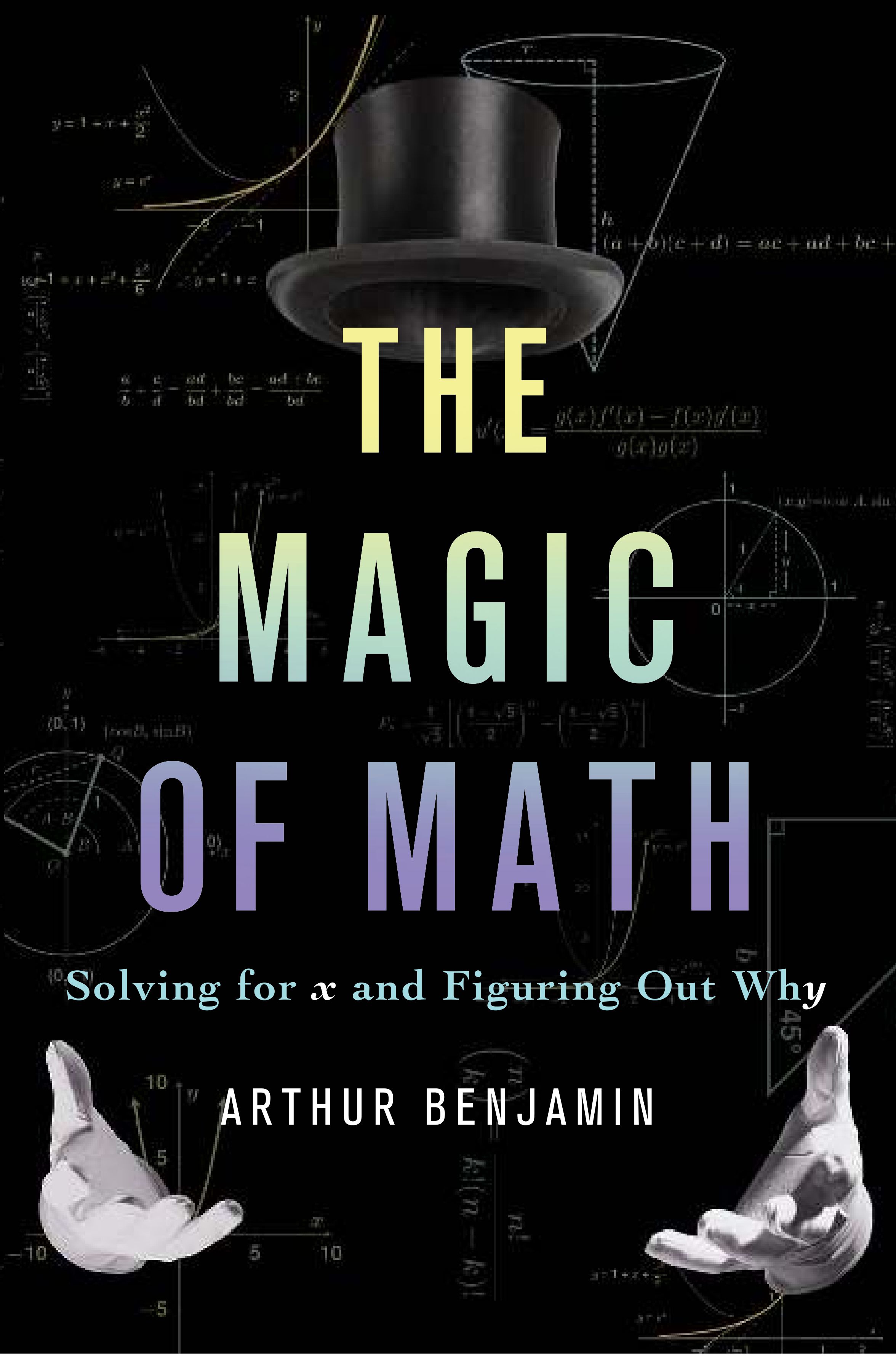
THE
MAGIC
OF MATH
Solving for x and Figuring Out Why
ARTHUR BENJAMIN

Advance Praise for *The Magic of Math*

"They say magicians should never reveal their secrets. Happily, Arthur Benjamin has ignored this silly adage—for in this small volume, Benjamin reveals to his audience the secrets of numbers and other mathematical illusions that have intrigued mathematicians for millennia."

—Edward B. Burger, president, Southwestern University, and author of *The 5 Elements of Effective Thinking*

"This book will be magical for my students, as it would have been for me throughout my school days. They'll be able to revisit the book frequently as they learn more math, finding deeper appreciation and discovering new areas to explore with each visit."

—Richard Rusczyk, founder, Art of Problem Solving, and director, USA Mathematical Talent Search

"In *The Magic of Math*, Arthur Benjamin has pulled off a seemingly impossible trick. He has made higher mathematics appear so natural and engaging that you will wonder why you were ever bored and confused in math class. There are many books that attempt to popularize mathematics. This is one of the best. On virtually every page I found myself learning new things, or looking at familiar topics in novel ways."

—Jason Rosenhouse, professor of Mathematics, James Madison University, and author of *The Monty Hall Problem*

"In *The Magic of Math*, mathemagician Arthur Benjamin gives us an entertaining and enlightening tour of a wide swath of fundamental mathematical ideas, presented in a way that is accessible to a broad audience. A particularly appealing feature of the book is the frequent use of friendly, down-to-earth explanations of the concepts and connections between them."

—Ronald Graham, president emeritus, American Mathematical Society, and coauthor of *Magical Mathematics*

"This book is a whirlwind tour of mathematics from arithmetic and algebra all the way to calculus and infinity, and especially the number *9*. Arthur Benjamin's enthusiastic and engaging writing style makes *The Magic of Math* a great addition to any math enthusiast's bag of tricks."

—Laura Taalman, professor of Mathematics and Statistics, James Madison University

"Mathematics is full of surprisingly beautiful patterns, which Arthur Benjamin's witty personality brings to life in *The Magic of Math*. You will not only discover many wonderful ideas, but you will also find some fun mathematical magic tricks that you will want to try out on your friends and family. Be prepared to learn that math is more entertaining than you may have thought."

—George W. Hart, mathematical sculptor, research professor, Stony Brook University, and cofounder, The Museum of Mathematics

"*The Magic of Math* is a delightful stroll through a garden filled with fascinating examples. Anyone with any interest in magic, puzzles, or math will have many hours of enjoyment in reading this book."

—Maria M. Klawe, president, Harvey Mudd College

"Arthur Benjamin has created an instant mathematical classic, by combining Isaac Asimov's clarity with Martin Gardner's taste and adding his own sense of fun and adventure. I wish he wrote this book when I was a kid."

—Paul A. Zeitz, professor and chair of Mathematics, University of San Francisco, and author of *The Art and Craft of Problem Solving*

"There's a playful joy to be found in this book, for readers at any level. Most magicians don't reveal their secrets, but in *The Magic of Math*, Arthur Benjamin shows how uncovering the mystery behind beautiful mathematical truths makes math even more marvelous to behold."

—Francis Su, president, Mathematical Association of America

"*The Magic of Math* offers an expansive, unforgettable journey through mathematics where numbers dance and mathematical secrets are revealed. Just open the book and start reading; you'll be swept over by the magic of Benjamin's writing. Luckily, there is no magician's code to these secrets as you'll undoubtedly want to share and perform them with family and friends."

—Tim Chartier, professor of Mathematics, Davidson College, and author of *Math Bytes*

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I dedicate this book to my wife, Deena,
and daughters, Laurel and Ariel.

Contents

CHAPTER ZERO

Intr0ducti0n

Throughout my life, I have always had a passion for magic. Whether I was watching other magicians or performing magic myself, I was fascinated with the methods used to accomplish amazing and impressive feats, and I loved learning its secrets. With just a handful of simple principles, I could even invent tricks of my own.

I had the same experience with mathematics. From a very early age, I saw that numbers had a magic all their own. Here's a trick you might enjoy. Think of a number between 20 and 100. Got it? Now add your digits together. Now subtract the total from your original number. Finally, add the digits of the new number together. Are you thinking of the number 9? (If not, you might want to check your previous calculation.) Pretty cool, huh? Mathematics is filled with magic like this, but most of us are never exposed to it in school. In this book, you will see how numbers, shapes, and pure logic can yield delightful surprises. And with just a little bit of algebra or geometry, you can often discover the secrets behind the magic, and perhaps even discover some beautiful mathematics of your own.

This book covers the essential mathematical subjects like numbers, algebra, geometry, trigonometry, and calculus, but it also covers topics that are not so well represented, like Pascal's triangle, infinity, and magical properties of numbers like 9, π, e, i, Fibonacci numbers, and the golden ratio. And although none of the big mathematical subjects can be completely covered in just a few dozen pages, I hope you come away with an understanding of the major concepts, a better idea of why they work, and an appreciation of the elegance and relevance of each subject. Even if you have seen some of these topics before, I hope you will see them and enjoy them with new perspectives. And as we learn more mathematics, the magic becomes more sophisticated and fascinating. For example, here is one of my favorite equations:

$$e^{i\pi} + 1 = 0$$

Some refer to this as "God's equation," because it uses the most important numbers in mathematics in one magical equation. Specifically, it uses 0 and 1, which are the foundations of arithmetic; $\pi = 3.14159\ldots$, which is the most important number in geometry; $e = 2.71828\ldots$, which is the most important number in calculus; and the *imaginary* number i, with a square of -1. We'll say more about π in Chapter 8, and the numbers i and e are described in greater detail in Chapter 10. In Chapter 11, we'll see the mathematics that help us understand this magical equation.

My target audience for this book is anyone who will someday need to take a math course, is currently taking a math course, or is finished taking math courses. In other words, I want this book to be enjoyed by everyone, from math-phobics to math-lovers. In order to do this, I need to establish some rules.

Rule 1: You can skip the gray boxes (except this one)!
Each chapter is filled with "Asides," where I like to go off on a tangent to talk about something interesting. It might be an extra example or a proof, or something that will appeal to the more advanced readers. You might want to skip these the first time you read this book (and maybe the second and third times too). And I do hope that you reread this book. Mathematics is a subject that is worth revisiting.

Rule 2: Don't be afraid to skip paragraphs, sections, or even chapters. In addition to skipping the gray boxes, feel free to go forward anytime you get stuck. Sometimes you need perspective on a topic before it fully sinks in. You will be surprised how much easier a topic can be when you come back to it later. It would be a shame to stop partway through the book and miss all the fun stuff that comes later.

Rule 3: Don't skip the last chapter. The last chapter, on the mathematics of infinity, has lots of mind-blowing ideas that they probably won't teach you in school, and many of these results do not rely on the earlier chapters. On the other hand, the last chapter does refer to ideas that appear in all of the previous chapters, so that might give you the extra incentive to go back and reread previous parts of the book.

Rule π: Expect the unexpected. While mathematics is a seriously important subject, it doesn't have to be taught in a serious and dry fashion. As a professor of mathematics at Harvey Mudd College, I can't resist the occasional pun, joke, poem, song, or magic trick to make a class more enjoyable, and they appear throughout these pages. And since this is a book, you don't have to hear me sing — lucky for you!

Follow these rules, and discover the magic of mathematics!

CHAPTER ONE

$1 + 2 + 3 + 4 + \cdots + 100 = 5050$

The Magic of Numbers

Number Patterns

The study of mathematics begins with numbers. In school, after we learn how to count and represent numbers using words or digits or physical objects, we spend many years manipulating numbers through addition, subtraction, multiplication, division, and other arithmetical procedures. And yet, we often don't get to see that numbers possess a magic of their own, capable of entertaining us, if we just look below the surface.

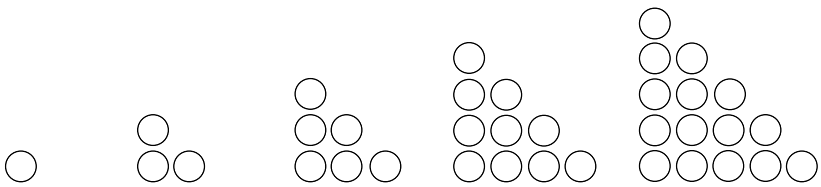
Let's start with a problem given to a mathematician named Karl Friedrich Gauss when he was just a boy. Gauss's teacher asked him and his classmates to add up all the numbers from 1 to 100, a tedious task designed to keep the students busy while the teacher did other work. Gauss astonished his teacher and classmates by immediately writing down the answer: 5050. How did he do it? Gauss imagined the numbers 1 through 100 split into two rows, with the numbers 1 through 50 on the top and the numbers 51 through 100 *written backward* on the bottom, as shown below. Gauss observed that each of the 50 columns would add up to the same sum, 101, and so their total would just be 50×101, which is 5050.

1	2	3	4	⋯	47	48	49	50
+ 100	+ 99	+ 98	+ 97	⋯	+ 54	+ 53	+ 52	+ 51
101	101	101	101	⋯	101	101	101	101

Splitting the numbers from 1 to 100 into two rows; each pair of numbers adds to 101

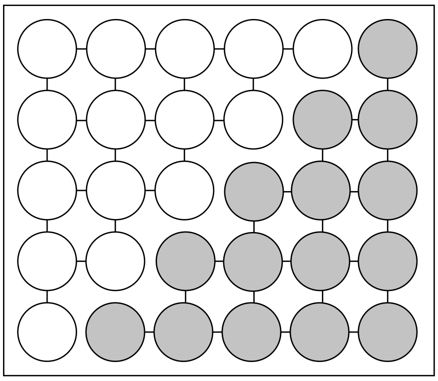
Gauss went on to become the greatest mathematician of the nineteenth century, not because he was quick at doing mental calculations, but because of his ability to make numbers dance. In this chapter, we will explore many interesting number patterns and start to see how numbers dance. Some of these patterns can be applied to do mental calculations more quickly, and some are just beautiful for their own sake.

We've used Gauss's logic to sum the first 100 numbers, but what if we wanted to sum 17 or 1000 or 1 million? We will, in fact, use his logic to sum the first n numbers, where n can be any number you want! Some people find numbers to be less abstract when they can visualize them. We call the numbers $1, 3, 6, 10$, and 15 *triangular numbers*, since we can create triangles like the ones below using those quantities of dots. (You might dispute that 1 dot forms a triangle, but nevertheless 1 is considered triangular.) The official definition is that the nth triangular number is $1 + 2 + 3 + \cdots + n$.



The first 5 triangular numbers are $1, 3, 6, 10$, and 15

Notice what happens when we put two triangles side by side, as depicted on the opposite page:



How many dots are in the rectangle?

Since the two triangles form a rectangle with 5 rows and 6 columns, there are 30 dots altogether. Hence, each original triangle must have half as many dots, namely 15. Of course, we knew that already, but the same argument shows that if you take two triangles with n rows and put them together as we did, then you form a rectangle with n rows and $n+1$ columns, which has $n \times (n+1)$ dots (often written more succinctly as $n(n+1)$ dots). As a result, we have derived the promised formula for **the sum of the first n numbers:**

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Notice what we just did: we saw a pattern to sum the first 100 numbers and were able to extend it to handle any problem of the same form. If we needed to add the numbers 1 through 1 million, we could do it in just two steps: multiply 1,000,000 by 1,000,001, then divide by 2!

Once you figure out one mathematical formula, other formulas often present themselves. For example, if we double both sides of the last equation, we get a formula for **the sum of the first n even numbers:**

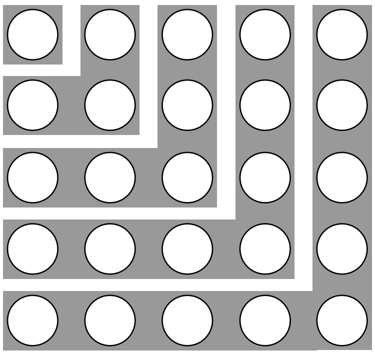
$$2 + 4 + 6 + \cdots + 2n = n(n+1)$$

What about **the sum of the first n odd numbers**? Let's look at what the numbers seem to be telling us.

$$\begin{array}{rcr} 1 & = & 1 \\ 1+3 & = & 4 \\ 1+3+5 & = & 9 \\ 1+3+5+7 & = & 16 \\ 1+3+5+7+9 & = & 25 \\ & \vdots & \end{array}$$

What is the sum of the first n odd numbers?

The numbers on the right are *perfect squares*: $1 \times 1, 2 \times 2, 3 \times 3$, and so on. It's hard to resist noticing the pattern that the sum of the first n odd numbers seems to be $n \times n$, often written as n^2. But how can we be sure that this is not just some temporary coincidence? We'll see a few ways to derive this formula in Chapter 6, but such a simple pattern should have a simple explanation. My favorite justification uses a count-the-dots strategy again, and reminds us of why we call numbers like 25 perfect *squares*. Why should the first 5 odd numbers add to 5^2? Just look at the picture of the 5-by-5 square below.



How many dots are in the square?

This square has $5 \times 5 = 25$ dots, but let's count the dots another way. Start with the 1 dot in the upper left corner. It is surrounded by 3 dots, then 5 dots, then 7 dots, then 9 dots. Consequently,

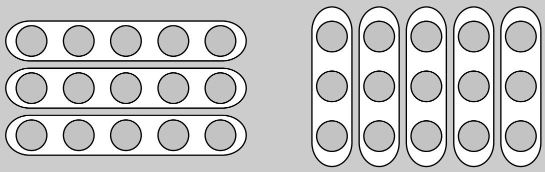
$$1+3+5+7+9=5^2$$

If we started with an n-by-n square, then we can break it into n (backward) L-shaped regions of sizes $1, 3, 5, \ldots, (2n-1)$. When viewed this way, we have a formula for **the sum of the first *n* odd numbers:**

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Aside

Later in this book, we'll see how the approach of counting dots (and the general approach of answering a question in two different ways) leads to some interesting results in advanced mathematics. But it can also be useful for understanding elementary mathematics as well. For example, *why does* $3 \times 5 = 5 \times 3$? I'm sure you haven't even questioned that statement since you were told, as a child, that the order of multiplication doesn't matter. (Mathematicians say that multiplication of numbers is *commutative*.) But why should 3 bags of 5 marbles contain the same amount as 5 bags of 3 marbles? The explanation is simple if you just count the dots in a 3-by-5 rectangle. Counting row by row, we see 3 rows of 5 dots apiece, giving us 3×5 dots. On the other hand, we also have 5 columns with 3 dots apiece, so there are also 5×3 dots.



Why does 3 × 5 = 5 × 3?

Let's apply the pattern from the sum of odd numbers to find an even more beautiful pattern. If our goal is to make the numbers dance, then you might say we are about to do some *square dancing*.

Consider this interesting pyramid of equations:

$$\begin{aligned} 1 + 2 &= 3 \\ 4 + 5 + 6 &= 7 + 8 \\ 9 + 10 + 11 + 12 &= 13 + 14 + 15 \\ 16 + 17 + 18 + 19 + 20 &= 21 + 22 + 23 + 24 \\ 25 + 26 + 27 + 28 + 29 + 30 &= 31 + 32 + 33 + 34 + 35 \\ &\vdots \end{aligned}$$

What patterns do you see? It's easy to count the numbers in each row: $3, 5, 7, 9, 11$, and so on. Next comes an unexpected pattern. What is the first number of each row? Judging from the first 5 rows, 1, 4, 9, 16, 25, . . . , they appear to be the perfect squares. Why is that? Let's look at the fifth row. How many numbers appear before row 5? If we count the numbers in the preceding four rows, we have $3+5+7+9$. To get the leading number of row 5, we just add 1 to this sum, so we really have the sum of the first 5 odd numbers, which we now know to be 5^2.

Now let's verify the fifth equation without actually adding any numbers. What would Gauss do? If we temporarily ignore the 25 at the beginning of the row, then there are 5 remaining numbers on the left, which are each 5 less than their corresponding numbers on the right.

25	26	27	28	29	30
	− 31	− 32	− 33	− 34	− 35
	−5	−5	−5	−5	−5

Comparing the left side of row 5 with the right side of row 5

Hence the five numbers on the right have a total that is 25 greater than their corresponding numbers on the left. But this is compensated for by the number 25 on the left. Hence the sums balance as promised. By the same logic, and a little bit of algebra, it can be shown that this pattern will continue indefinitely.

Aside

For those who wish to see the little bit of algebra now, here it is. Row n is preceded by $3+5+7+\cdots+(2n-1) = n^2-1$ numbers, so the left side of the equation must start with the number n^2, followed by the next n consecutive numbers, n^2+1 through n^2+n. The right side has n consecutive numbers starting with n^2+n+1 through n^2+2n. If we temporarily ignore the n^2 number on the left, we see that the n numbers on the right are each n larger than their corresponding numbers on the left, so their difference is $n \times n$, which is n^2. But this is compensated for on the left by the initial n^2 term, so the equations balance.

Time for a new pattern. We saw that odd numbers could be used to make squares. Now let's see what happens when we put all the odd numbers in one big triangle, as shown on the next page.

We see that $3+5=8$, $7+9+11=27$, $13+15+17+19=64$. What do the numbers 1, 8, 27, and 64 have in common? They are perfect cubes! For example, summing the five numbers in the fifth row, we get

$$
\begin{array}{ccccccccccccccc}
 & & & & 1 & & & & & = & 1 & = & 1^3 \\
 & & & 3 & + & 5 & & & & = & 8 & = & 2^3 \\
 & & 7 & + & 9 & + & 11 & & & = & 27 & = & 3^3 \\
 & 13 & + & 15 & + & 17 & + & 19 & & = & 64 & = & 4^3 \\
21 & + & 23 & + & 25 & + & 27 & + & 29 & = & 125 & = & 5^3 \\
 & & & & \vdots & & & & & & \vdots & & \vdots
\end{array}
$$

An odd triangle

$$21 + 23 + 25 + 27 + 29 = 125 = 5 \times 5 \times 5 = 5^3$$

The pattern seems to suggest that the sum of the numbers in the nth row is n^3. Will this always be the case, or is it just some *odd* coincidence? To help us understand this pattern, check out the middle numbers in rows 1, 3, and 5. What do you see? The perfect squares 1, 9, and 25. Rows 2 and 4 don't have middle numbers, but surrounding the middle are the numbers 3 and 5 with an average of 4, and the numbers 15 and 17 with an average of 16. Let's see how we can exploit this pattern.

Look again at row 5. Notice that we can *see* that the sum is 5^3 without actually adding the numbers by noticing that these five numbers are symmetrically centered around the number 25. Since the average of these five numbers is 5^2, then their total must be $5^2 + 5^2 + 5^2 + 5^2 + 5^2 = 5 \times 5^2$, which is 5^3. Similarly, the average of the four numbers of row 4 is 4^2, so their total must be 4^3. With a little bit of algebra (which we won't do here), you can show that the average of the n numbers in row n is n^2, so their total must be n^3, as desired.

Since we're talking about cubes and squares, I can't resist showing you one more pattern. What totals do you get as you add the cubes of numbers starting with 1^3?

$$
\begin{array}{l}
1^3 = 1 = \mathbf{1}^2 \\
1^3 + 2^3 = 9 = \mathbf{3}^2 \\
1^3 + 2^3 + 3^3 = 36 = \mathbf{6}^2 \\
1^3 + 2^3 + 3^3 + 4^3 = 100 = \mathbf{10}^2 \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225 = \mathbf{15}^2 \\
\qquad\qquad\qquad\vdots
\end{array}
$$

The sum of the cubes is always a perfect square

When we start summing cubes, we get the totals $1, 9, 36, 100, 225$, and so on, which are all perfect squares. But they're not just *any* perfect squares; they are the squares of $1, 3, 6, 10, 15$, and so on, which are all triangular numbers! Earlier we saw that these were the sums of integers and so, for example,

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225 = 15^2 = (1 + 2 + 3 + 4 + 5)^2$$

To put it another way, the sum of the cubes of the first n numbers is the square of the sum of the first n numbers. We're not quite ready to prove that result now, but we will see two proofs of this in Chapter 6.

Fast Mental Calculations

Some people look at these number patterns and say, "Okay, that's nice. But what good are they?" Most mathematicians would probably respond like any artist would—by saying that a beautiful pattern needs no justification other than its beauty. And the patterns become even more beautiful the more deeply we understand them. But sometimes the patterns can lead to real applications.

Here's a simple pattern that I had the pleasure of discovering (even if I wasn't the first person to do so) when I was young. I was looking at pairs of numbers that added up to 20 (such as 10 and 10, or 9 and 11), and I wondered how large the product could get. It seemed that the largest product would occur when both numbers were equal to 10, and the pattern confirmed that.

	Distance Below 100
$10 \times 10 = 100$	
$9 \times 11 = 99$	1
$8 \times 12 = 96$	4
$7 \times 13 = 91$	9
$6 \times 14 = 84$	16
$5 \times 15 = 75$	25
$\vdots$	$\vdots$

The product of numbers that add to 20

The pattern was unmistakable. As the numbers were pulled farther apart, the product became smaller. And how far below 100 were they?

$1, 4, 9, 16, 25, \ldots$, which were $1^2, 2^2, 3^2, 4^2, 5^2$, and so on. Does this pattern always work? I decided to try another example, by looking at pairs of numbers that add up to 26.

	Distance Below 169
$13 \times 13 = 169$	
$12 \times 14 = 168$	1
$11 \times 15 = 165$	4
$\boxed{10 \times 16 = 160}$	$\boxed{9}$
$9 \times 17 = 153$	16
$8 \times 18 = 144$	25
$\vdots$	$\vdots$

The product of numbers that add to 26

Once again, the product was maximized when we chose the two numbers to be equal, and then the product decreased from 169 by 1, then 4, then 9, and so on. After a few more examples, I was convinced that the pattern was true. (I'll show you the algebra behind it later.) Then I saw a way that this pattern could be applied to squaring numbers faster.

Suppose we want to square the number 13. Instead of performing 13×13 directly, we will perform the easier calculation of $10 \times 16 = 160$. This is almost the answer, but since we went up and down 3, it is shy of the answer by 3^2. Thus,

$$13^2 = (10 \times 16) + 3^2 = 160 + 9 = 169$$

Let's try another example. Try doing 98×98 using this method. To do this, we go up 2 to 100, then down 2 to 96, then add 2^2. That is,

$$98^2 = (100 \times 96) + 2^2 = 9600 + 4 = 9604$$

Squaring numbers that end in 5 are especially easy, since when you go up and down 5, the numbers you are multiplying will both end in 0. For example,

$$35^2 = (30 \times 40) + 5^2 = 1200 + 25 = 1225$$

$$55^2 = (50 \times 60) + 5^2 = 3000 + 25 = 3025$$

$$85^2 = (80 \times 90) + 5^2 = 7200 + 25 = 7225$$