Understanding Complex Systems

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Networks of Networks: The Last Frontier of Complexity

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Understanding Complex Systems

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Sometimes life is complicate, sometimes it is just complex

Preface

Modern life in fully developed countries relies on the coordinated functioning of several infrastructures such as Electric System, Aqueducts, Communication Assets, Fresh food distribution chains, Gas-ducts, Oil Pipelines, Transports, Financial networks, etc. Several of such infrastructures have been regarded as critical since they provide vital services to sustain the modern technological society and its development.

During the last decades, the level of awareness about the importance of protecting our *Critical Infrastructures* (CIs) has been steadily growing. In this respect, US has been the first country to take an official financial commitment by means of the celebrated American Presidential Directive PDD-63 of May 1998 under the Clinton administration. After ten years also the European Community made a similar commitment through the EUDIR Council Directive 2008/114/EC (dated December the 8-th, 2008), that has been afterwards implemented by the EU member states. It has to be noticed that, while the US directive is very broad in its scope, the EU directive is presently limited to the energetic, transport and financial sectors.

The functioning of Critical Infrastructures requires both physical components and human actors. It is therefore important not only to employ reliable components, but also to understand human behaviour at both individual and collective levels. Moreover, each infrastructure resorts to other CIs (typically, but not limited to, energy and ICT) to accomplish its goals: in other words, CIs are *inter-dependent*. Identifying, understanding and analysing critical infrastructure interdependencies is therefore a crucial task to be pursued by the scientific community at both the academic and applied levels [1].

In the development of CIs, the ICT sector has played a crucial role in several respects. ICT pervades any complex activity of modern societies based on communications and represents a fundamental part for the governance of any complex infrastructure. The quality and quantity of information-based services provided to our modern society has been steadily increasing during last 30 years (Web, e-mail, e-commerce, social networking, e-banking, e-health, Web-based entertainment, SCADA systems, etc). In order to improve their performance and to enhance their reliability, the infrastructures have been endowed with increasingly complex connection networks and computerized systems, thus allowing their governance optimization and reducing the humans allocated to that purpose. Nowadays, our society is on the verge of a new revolution in which the infrastructures are required to become *smart* and to integrate into a *smart* technological environment. Driving the advent of a *smart* society on a painless and secure path represents one of the most difficult challenges for all the technologically advanced countries.

Most of the infrastructures exhibit a network structure. In the last decade, stemming from the availability of large data and based on the statistical physicist perspective of the graph theory, a new paradigm to describe large networks has blossomed: the Network Science [2, 3].

Network Science has revealed a powerful and unifying tool that enables to treat on the same footing widely different networked systems, ranging from biology to sociology to power grids to the Internet and the World Wide Web. In fact, despite their intrinsic differences, all such networks are large systems consisting of simple elementary units (the nodes) interacting via basic mechanisms (represented by the links). Statistical Physics teaches us that this is the case where to expect the occurrence of *emergent behaviours*, i.e., of collective (systemic) effects. In fact, while each component may be perfectly working, the system as a whole can be in a failure state: as an example, think about a big traffic jam, where all the cars, lights, indications, navigators and roads are perfectly functioning and yet everybody is stuck.

Financial networks' analysis represented the forerunner to assess the concept of systemic risk in real infrastructures. Nowadays, several financial institutions consider and employ the global metrics developed by EU network scientists [4] to assess their risk level and robustness consistently with the Basel III Stress Testing [5].

Applying the Network Science paradigm to Inter-dependent Critical Infrastructures has lead to the development of the concept of "Networks of Networks": the *NetONets*. While from the graph-theory point of view a network of networks is just a larger (inhomogeneous) network, in real life infrastructural networks are governed and operated separately and interactions are only allowed at well-defined boundaries. Assessing properties on NetONets instead of that on single networks is like deciding to consider males and females instead of human beings as a single community: depending on the question to answer, either approach may be the most fruitful.

The first applications of the *NetONets* approach to understand critical infrastructures has been related to the propagation of failures in inter-dependent infrastructures modelled as either trees or planar lattices [6, 7]. However, the upheaval of the interest in *NetONets* has followed the publication of a Nature paper on a percolation model of cascade failures in coupled ICT/power networks [8]. Another important step towards real applications has been the analysis of the North America inter-connected electric systems [9] aiming to reduce the global vulnerability of the system.

In general, numerous efforts are nowadays devoted to develope the mathematics of *NetONets*. While most of the current models have a percolative flavour [10–13], some new directions in understanding the dynamics on *NetONets* are being explored [14–16] resorting to the spectral properties of networks. The European efforts on the subject have recently concentrated in the "MULTIPLEX" project [17] combining top scientists in Complexity and Algorithmic. While the complexity approach allows to concentrate on systemic effects and emergent behavior, other routes have to be considered to perform the analyses of the systems needed for several tasks including management, planning the development, enhancing the security, defining coordinated national and EU/US contingency plans, and assessing the policies at the state and the regional levels. To such an aim, several techniques such as I/O models, federated simulations, agent-based models, time-series analysis are employed. Each of the previous approaches provides a partial perspective of the system behaviour; however to manage and understand the complexity of our society, all of them are required. Our book aims to foster a meta-community able to share and integrate all such perspectives.

This volume is structured along three main sections: part I covers the theoretical approaches, part II provides some applications and part III is devoted to phenomenological modelling. The former taxonomy has been mainly introduced for the sake of presentation. However, due to their inter-disciplinarity, it is difficult to ascribe each contribution to a specific topic only. To improve legibility, each part of the book is endowed with a brief overview of its contents.

We have spent our best efforts to provide the reader with as different contributions as possible; most of the authors have been actively involved in the *NetONets* and related conference series. However, by no means our book can be regarded as exhaustive. Probably, the I/O models [18] represent the most significant lack in our book. Some of the most important topics, such as the systemic risk analysis [19] or time series analysis, would have deserved a more extended treatment. We hope to be able to cover such topics in a nearby future.

Furthermore, there are important topics that are crucial to develop in the nearby future. In particular, the human behaviour, both at the management and at the end user levels, must be accounted for improving the analysis, modelling and simulation of inter-dependent infrastructures. Regarding the complexity approach, it is crucial to build up methodological tools for the statistical analysis of 'small' systems. In fact, while most of the current techniques are aimed to understand the behaviour of the system in the infinite-size limit, almost all infrastructural networks exhibit a relatively small size.

We have tried hard to produce a book that could be regarded as an updated reference for the *NetONets* state-of-the-art. To the same purpose of providing updated information, we have also built a website (netonets.org) wherein to gather and advertise all the initiatives in the field.

One of the main barriers to overcome is the lack of a common language. It is therefore crucial to foster the up-growing *NetONets* community providing a common ground for knowledge sharing. We hope that our efforts will contribute to such a direction.

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Part I
Theoretical Approaches

This part of the book is devoted to the theoretical approaches to interdependent networks. The state of the art of such a novel and dynamic field is experiencing a continuous growth. Here we have selected, mainly for historical reasons, the contributions stemming from the Statistical Physics approach.

Modelling interdependent networks consists in defining different graphs and the interactions among them. In the multiplex approach, the different layers are modelled by means of different types of links. In the interacting networks approach, the different layers are explicitly modelled as separate networks and the links among them represent the inter-layer interactions.

In Chaps. 1–3 authors rely on 'static' approaches aimed at assessing the robustness and/or the resilience of interdependent systems upon both random failures and targeted attacks. Considering the dynamics of the systems upon continuous stressing leads to the introduction of further effects discussed in Chaps. 4 and 5

Chapter 1
Network of Interdependent Networks: Overview of Theory and Applications

Dror Y. Kenett, Jianxi Gao, Xuqing Huang, Shuai Shao, Irena Vodenska, Sergey V. Buldyrev, Gerald Paul, H. Eugene Stanley and Shlomo Havlin

Abstract Complex networks appear in almost every aspect of science and technology. Previous work in network theory has focused primarily on analyzing single networks that do not interact with other networks, despite the fact that many real-world networks interact with and depend on each other. Very recently an analytical

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framework for studying the percolation properties of interacting networks has been introduced. Here we review the analytical framework and the results for percolation laws for a network of networks (NON) formed by n interdependent random networks. The percolation properties of a network of networks differ greatly from those of single isolated networks. In particular, although networks with broad degree distributions, e.g., scale-free networks, are robust when analyzed as single networks, they become vulnerable in a NON. Moreover, because the constituent networks of a NON are connected by node dependencies, a NON is subject to cascading failure. When there is strong interdependent coupling between networks, the percolation transition is discontinuous (is a first-order transition), unlike the well-known continuous second-order transition in single isolated networks. We also review some possible real-world applications of NON theory.

1.1 Introduction

The interdisciplinary field of network science has attracted great attention in recent years [1–26]. This has taken place because an enormous amount of data regarding social, economic, engineering, and biological systems has become available over the past two decades as a result of the information and communication revolution brought about by the rapid increase in computing power. The investigation and growing understanding of this extraordinary amount of data will enable us to make the infrastructures we use in everyday life more efficient and more robust. The original model of networks, random graph theory, developed in the 1960s by Erdős and Rényi (ER), is based on the assumption that every pair of nodes is randomly connected with the same probability (leading to a Poisson degree distribution). In parallel, lattice networks in which each node has the same number of links have been used in physics to model physical systems. While graph theory was a well-established tool in the mathematics and computer science literature, it could not adequately describe modern, real-world networks. Indeed, the pioneering observation by Barabási in 1999 [2], that many real networks do not follow the ER model but that organizational principles naturally arise in most systems, led to an overwhelming accumulation of supporting data, new models, and novel computational and analytical results, and led to the emergence of a new science: complex networks.

Significant advances in understanding the structure and function of networks, and mathematical models of networks have been achieved in the past few years. These are now widely used to describe a broad range of complex systems, from techno-social systems to interactions amongst proteins. A large number of new measures and methods have been developed to characterize network properties, including measures of node clustering, network modularity, correlation between degrees of neighboring nodes, measures of node importance, and methods for the identification and extraction of community structures. These measures demonstrated that many real networks, and in particular biological networks, contain network motifs—small specific subnetworks—that occur repeatedly and provide information about functionality [8]. Dynamical processes, such as flow and electrical transport in heterogeneous networks, were shown to be significantly more efficient compared to ER networks [27, 28].

Complex networks are usually non-homogeneous structures that exhibit a power-law form in their degree (number of links per node) distribution. These systems are called scale-free networks. Some examples of real-world scale-free networks include the Internet [3], the WWW [4], social networks representing the relations between individuals, infrastructure networks such as airlines [29, 30], networks in biology, in particular networks of protein-protein interactions [31], gene regulation, and biochemical pathways, and networks in physics, such as polymer networks or the potential energy landscape network. The discovery of scale-free networks has led to a re-evaluation of the basic properties of networks, such as their robustness, which exhibit a character that differs drastically from that of ER networks. For example, while homogeneous ER networks are vulnerable to random failures, heterogeneous scale-free networks are extremely robust [4, 5]. An important property of these infrastructures is their stability, and it is thus important that we understand and quantify their robustness in terms of node and link functionality. Percolation theory was introduced to study network stability and to predict the critical percolation threshold [5]. The robustness of a network is usually (i) characterized by the value of the critical threshold analyzed using percolation theory [32] or (ii) defined as the integrated size of the largest connected cluster during the entire attack process [33]. The percolation approach was also extremely useful in addressing other scenarios, such as efficient attacks or immunization [6, 7, 14, 34, 35], for obtaining optimal path [36] as well as for designing robust networks [33]. Network concepts were also useful in the analysis and understanding of the spread of epidemics [37, 38], and the organizational laws of social interactions, such as friendships [39, 40] or scientific collaborations [41]. Moreira et al. investigated topologically-biased failure in scale-free networks and controlled the robustness or fragility by fine-tuning the topological bias during the failure process [42].

Because current methods deal almost exclusively with individual networks treated as isolated systems, many challenges remain [43]. In most real-world systems an individual network is one component within a much larger complex multi-level network (is part of a network of networks). As technology has advanced, coupling between networks has become increasingly strong. Node failures in one network will cause the failure of dependent nodes in other network, and vice-versa [44]. This recursive process can lead to a cascade of failures throughout the network of networks system. The study of individual particles has enabled physicists to understand the properties of a gas, but in order to understand and describe a liquid or a solid the interactions between the particles also need to be understood. So also in network theory, the study of isolated single networks brings extremely limited results—real-world noninteracting systems are extremely rare in both classical physics and network study. Most real-world network systems continuously interact with other networks, especially since modern technology has accelerated network interdependency.

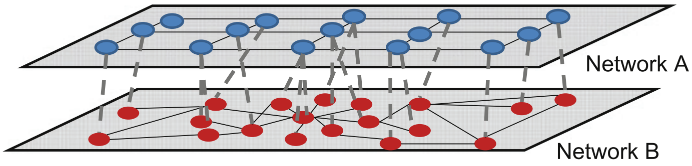

Fig. 1.1 Example of two interdependent networks. Nodes in network B (communications network) are dependent on nodes in network A (power grid) for power; nodes in network A are dependent on network B for control information

To adequately model most real-world systems, understanding the interdependence of networks and the effect of this interdependence on the structural and functional behavior of the coupled system is crucial. Introducing coupling between networks is analogous to the introduction of interactions between particles in statistical physics, which allowed physicists to understand the cooperative behavior of such rich phenomena as phase transitions. Surprisingly, preliminary results on mathematical models [44, 45] show that analyzing complex systems as a network of coupled networks may alter the basic assumptions that network theory has relied on for single networks. Here we will review the main features of the theoretical framework of Network of Networks (NON), and present some real world applications.

1.2 Overview

In order to model interdependent networks, we consider two networks, A and B, in which the functionality of a node in network A is dependent upon the functionality of one or more nodes in network B (see Fig. 1.1), and vice-versa: the functionality of a node in network B is dependent upon the functionality of one or more nodes in network A. The networks can be interconnected in several ways. In the most general case we specify a number of links that arbitrarily connect pairs of nodes across networks A and B. The direction of a link specifies the dependency of the nodes it connects, i.e., link $A_i \rightarrow B_j$ provides a critical resource from node A_i to node B_j. If node A_i stops functioning due to attack or failure, node B_j stops functioning as well but not vice-versa. Analogously, link $B_i \rightarrow A_j$ provides a critical resource from node B_i to node A_j.

To study the robustness of interdependent networks systems, we begin by removing a fraction $1 - p$ of network A nodes and all the A-edges connected to these nodes. As an outcome, all the nodes in network B that are connected to the removed A-nodes by $A \rightarrow B$ links are also removed since they depend on the removed nodes in network A. Their B edges are also removed. Further, the removed B nodes will cause the removal of additional nodes in network A which are connected to the removed B-nodes by $B \rightarrow A$ links. As a result, a cascade of failures that eliminates virtually all nodes in both networks can occur. As nodes and edges are removed, each network breaks up into connected components (clusters). The clusters in network A (connected by A-edges) and the clusters in network B (connected by B-edges) are different since the networks are each connected differently. If one assumes that small clusters (whose size is below certain threshold) become non-functional, this may invoke a recursive process of failures that we now formally describe.

Our insight based on percolation theory is that when the network is fragmented the nodes belonging to the giant component connecting a finite fraction of the network are still functional, but the nodes that are part of the remaining small clusters become non-functional. Thus in interdependent networks only the giant mutually-connected cluster is of interest. Unlike clusters in regular percolation whose size distribution is a power law with a p-dependent cutoff, at the final stage of the cascading failure process just described only a large number of small mutual clusters and one giant mutual cluster are evident. This is the case because the probability that two nodes that are connected by an A-link and their corresponding two nodes are also connected by a B-link scales as $1/N_{\mathrm{B}}$, where N_{B} is the number of nodes in network B. So the centrality of the giant mutually-connected cluster emerges naturally and the mutual giant component plays a prominent role in the functioning of interdependent networks. When it exists, the networks preserve their functionality, and when it does not exist, the networks split into fragments so small they cannot function on their own.

We ask three questions: What is the critical $p = p_c$ below which the size of any mutual cluster constitutes an infinitesimal fraction of the network, i.e., no mutual giant component can exist? What is the fraction of nodes $P_\infty(p)$ in the mutual giant component at a given p? How do the cascade failures at each step damage the giant functional component?

Note that the problem of interacting networks is complex and may be strongly affected by variants in the model, in particular by how networks and dependency links are characterized. In the following section we describe several of these model variants.

1.3 Theory of Interdependent Networks

In order to better understand how present-day crucially-important infrastructures interact, Buldyrev et al. [44] recently developed a mathematical framework to study percolation in a system of two coupled interdependent networks subject to cascading failure. Their analytical framework is based on a generating function formalism widely used in studies of single-network percolation and single-network structure [41, 44, 46]. Using the framework to study interdependent networks, we can follow the dynamics of the cascading failures as well as derive analytic solutions for the final steady state. Buldyrev et al. [44] found that interdependent networks were significantly more vulnerable than their noninteracting counterparts. The failure of even a small number of elements within a single network in a system may trigger a catastrophic cascade of events that propagates across the global connectivity. For a

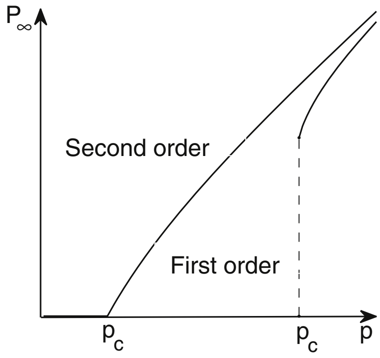

Fig. 1.2 Schematic demonstration of first and second order percolation transitions. In the second order case, the giant component is continuously approaching zero at the percolation threshold $p = p_c$. In the first order case the giant component approaches zero discontinuously. After [47]

fully coupled case in which each node in one network depends on a functioning node in another network and vice versa, Buldyrev et al. [44] found a first-order discontinuous phase transition, which differs significantly from the second-order continuous phase transition found in single isolated networks (Fig. 1.2). This interesting phenomenon is caused by the presence of two types of links: (i) connectivity links within each network and (ii) dependency links between networks. Parshani et al. [45] showed that, when the dependency coupling between the networks is reduced, at a critical coupling strength the percolation transition becomes second-order.

We now present the theoretical methodology used to investigate networks of interdependent networks (see Ref. [47]), and provide examples from different classes of networks.

1.3.1 Generating Functions for a Single Network

We begin by describing the generating function formalism for a single network that is also useful when studying interdependent networks. Here we assume that all N_i nodes in network i are randomly assigned a degree k from a probability distribution $P_i(k)$, and are randomly connected, the only constraint being that the node with degree k has exactly k links [48]. We define the generating function of the degree distribution

$$G_i(x) \equiv \sum_{k=0}^{\infty} P_i(k)x^k, \tag{1.1}$$

where x is an arbitrary complex variable. The average degree of network i is

$$\langle k \rangle_i = \sum_{k=0}^{\infty} k P_i(k) = \left.\frac{\partial G_i}{\partial x}\right|_{x \to 1} = G_i'(1). \tag{1.2}$$

In the limit of infinitely large networks $N_i \to \infty$, the random connection process can be modeled as a branching process in which an outgoing link of any node has a probability $kP_i(k)/\langle k\rangle_i$ of being connected to a node with degree k, which in turn has $k-1$ outgoing links. The generating function of this branching process is defined as

$$H_i(x) \equiv \frac{\sum_{k=0}^{\infty} P_i(k)kx^{k-1}}{\langle k\rangle_i} = \frac{G_i'(x)}{G_i'(1)}. \tag{1.3}$$

The probability f_i that a randomly chosen outgoing link does not lead to an infinitely large giant component satisfies a recursive relation $f_i = H_i(f_i)$. Accordingly, the probability that a randomly chosen node does belong to a giant component is given by $g_i = G_i(f_i)$. Once a fraction $1-p$ of nodes is randomly removed from a network, its generating function remains the same, but must be computed from a new argument $z \equiv px + 1 - p$ [46]. Thus $P_{\infty,i}$, the fraction of nodes that belongs to the giant component, is given by [46],

$$P_{\infty,i} = pg_i(p), \tag{1.4}$$

where

$$g_i(p) = 1 - G_i[pf_i(p) + 1 - p], \tag{1.5}$$

and $f_i(p)$ satisfies

$$f_i(p) = H_i[pf_i(p) + 1 - p]. \tag{1.6}$$

As p decreases, the nontrivial solution $f_i < 1$ of Eq. (1.6) gradually approaches the trivial solution $f_i = 1$. Accordingly, $P_{\infty,i}$—selected as an order parameter of the transition—gradually approaches zero as in the second-order phase transition and becomes zero when two solutions of Eq. (1.6) coincide at $p = p_c$. At this point the straight line corresponding to the right hand side of Eq. (1.6) becomes tangent to the curve corresponding to its left hand side, yielding

$$p_c = 1/H'_i(1). \tag{1.7}$$

For example, for Erdős-Rényi (ER) networks [49–51], characterized by the Poisson degree distribution,

$$G_i(x) = H_i(x) = \exp[\langle k\rangle_i(x-1)], \tag{1.8}$$

$$g_i(p) = 1 - f_i(p), \tag{1.9}$$

$$f_i(p) = \exp\{p\langle k\rangle_i[f_i(p) - 1]\}, \tag{1.10}$$

and

$$p_c = \frac{1}{\langle k\rangle_i}. \tag{1.11}$$

Finally, using Eqs. (1.4), (1.9), and (1.10), one obtains a direct equation for $P_{\infty,i}$

$$P_{\infty,i} = p[1 - \exp(-\langle k \rangle_i P_{\infty,i})]. \quad (1.12)$$

1.3.2 Two Networks with One-to-One Correspondence of Interdependent Nodes

To initiate an investigation of the multitude of problems associated with interacting networks, Buldyrev et al. [44] restricted themselves to the case of two randomly and independently connected networks with the same number of nodes, specified by their degree distributions $P_A(k)$ and $P_B(k)$. They also assumed every node in the two networks to have one B $\rightarrow$ A link and one A $\rightarrow$ B link connecting the same pair of nodes, i.e., the dependencies between networks A and B establish a isomorphism between them that allows us to assume that nodes in A and B coincide (e.g., are at the same corresponding geographic location—if a node in network A fails, the corresponding node in network B also fails, and vice versa). We also assume, however, that the A-edges and B-edges in the two networks are independent.

Unlike the percolation transition in a single network, the mutual percolation transition in this model is a first-order phase transition at which the order parameter (i.e., the fraction of nodes in the mutual giant component) abruptly drops from a finite value at $p_c + \varepsilon$ to zero at $p_c - \varepsilon$. Here ε is a small number that vanishes as the size of network increases $N \rightarrow \infty$. In this range of p, a removal of single critical node may lead to a complete collapse of a seemingly robust network. The size of the largest component drops from $N P_\infty$ to a small value, which rarely exceeds 2.

Note that the value of p_c is significantly larger than in single-network percolation. In two interdependent ER networks, for example, $p_c = 2.4554/\langle k \rangle$, while in a single network, $p_c = 1/\langle k \rangle$. For two interdependent scale-free networks with a power-law degree distribution $P_A(k) \sim k^{-\lambda}$, the mutual percolation threshold is $p_c > 0$, even for $2 < \lambda \leq 3$, when the percolation threshold in a single network is zero.

Note also that, in this new model, networks with a broader degree distribution are less robust against random attack than networks having a narrower degree distribution but the same average degree. This behavior also differs from that found in single networks. To understand this we note that (i) in interdependent networks, nodes are randomly connected—high degree nodes in one network can connect to low degree nodes in other networks, and (ii) at each time step, failing nodes in one network cause their corresponding nodes (and their edges) in the other network to also fail. Thus although hubs in single networks strongly contribute to network robustness, in interdependent networks they are vulnerable to cascading failure. If a network has a fixed average degree, a broader distribution means more nodes with low degree to balance the high degree nodes. Since the low degree nodes are more easily disconnected the advantage of a broad distribution in single networks becomes a disadvantage in interdependent networks.

The following features have been investigated analytically in Ref. [52], a study that assumes that the degrees of the interdependent nodes exactly coincide, but that both networks are randomly and independently connected by their connectivity links. Reference [52] shows that, for two networks with the same degree distribution $P_{\rm A}(k)$ of connectivity links and random dependency links, studied in Ref. [44], the fraction of nodes in the giant component is

$$P_\infty = p[1 - G_{\rm A}(z)]^2, \tag{1.13}$$

where $0 \le z \le 1$ is a new variable $z = 1 - p + pf_{\rm A}$ satisfying equation

$$\frac{[1 - H_{\rm A}(z)][1 - G_{\rm A}(z)]}{1 - z} = \frac{1}{p}. \tag{1.14}$$

while in case of coinciding degrees of interdependent nodes Eqs. (1.13) and (1.14) become respectively

$$P_\infty = p[1 - 2G_{\rm A}(z) + G_{\rm A}(z^2)] \tag{1.15}$$

and

$$\frac{1 - (1 + z)H_{\rm A}(z) + zH_{\rm A}(z^2)}{1 - z} = \frac{1}{p}. \tag{1.16}$$

The left-hand side of Eq. (1.14) always has a single maximum at $0 < z_c < 0$, and the solution abruptly disappear if p becomes less than p_c, the inverse left hand side at z_c. This situation corresponds to the first order transition. In contrast, the left-hand side of Eq. (1.16) has a maximum only if $H'_{\rm A}(1)$ converges, which corresponds to $\lambda > 3$ when there is a power law tail in the degree distribution. In this case, p_c is the inverse maximum value of the left-hand side of Eq. (1.16), e.g., for ER networks, $p_c = 1.7065/\langle k \rangle$. When $\lambda < 3$, $H'(z)$ diverges for $z \to 1$ and $p_c = 0$, $P_\infty = 0$ as in the case of regular percolation on a single network, for which Eqs. (1.4), (1.5), and (1.6) give

$$P_\infty = p[1 - G_{\rm A}(z)], \tag{1.17}$$

and

$$\frac{1 - H_{\rm A}(z)}{1 - z} = \frac{1}{p}. \tag{1.18}$$

Thus for networks with coinciding degrees of the interdependent nodes for $\lambda < 3$, the transition becomes a second-order transition with $p_c = 0$. In the marginal case of $\lambda = 3$, $p_c > 0$, but the transition is second-order.

From Eqs. (1.13)–(1.18) it follows that, if $H'_{\rm A}(1)$ converges, the networks with coinciding degrees of interdependent nodes are still less robust than single networks, still undergo collapse via a first-order phase transition, but are always more robust than networks with uncorrelated degrees of interdependent nodes. If the average degree is fixed, the robustness of the networks with coinciding degrees of interdependent nodes increases as the degree distribution broadens in the same way as for single networks. Similar observations have been made in numerical studies of interdependent networks with correlated degrees of interdependent nodes [53]. In conclusion, the robustness of interdependent networks increases if the degrees of the interdependent nodes are correlated, i.e., if the hubs are more likely to depend on hubs than on low-degree nodes. For the case of common connectivity links in both networks see Dong et al. [54] and Cellai et al. [55].

1.3.3 Framework of Two Partially Interdependent Networks

A generalization of the percolation theory for two fully interdependent networks was developed by Parshani et al. [45], who studied a more realistic case of a pair of partially-interdependent networks. Here both interacting networks have a certain fraction of completely autonomous nodes whose function does not directly depend on nodes in the other network. They found that when the fraction of autonomous nodes increases above a certain threshold, the collapse of the interdependent networks characterized by a first-order transition observed in Ref. [44] changes, at a critical coupling strength, to a continuous second-order transition as in classical percolation theory [32].

We now describe in more detail the framework developed in [45]. This framework consists of two networks A and B with the number of nodes N_{A} and N_{B}, respectively. Within network A, the nodes are randomly connected by A edges with degree distribution $P_{\mathrm{A}}(k)$, and the nodes in network B are randomly connected by B edges with degree distribution $P_{\mathrm{B}}(k)$. In addition, a fraction q_{A} of network A nodes depends on the nodes in network B and a fraction q_{B} of network B nodes depends on the nodes in network A. We assume that a node from one network depends on no more than one node from the other network, and if A_i depends on B_j, and B_j depends on A_k, then $k = i$. The latter "no-feedback" condition (see Fig. 1.3) disallows configurations that can collapse without taking into account their internal connectivity [56]. Suppose that the initial removal of nodes from network A is a fraction $1 - p$.

We next present the formalism for the cascade process, step by step (see Fig. 1.4). The remaining fraction of network A nodes after an initial removal of nodes is $\psi'_1 \equiv p$. The initial removal of nodes disconnects some nodes from the giant component. The remaining functional part of network A thus contains a fraction $\psi_1 = \psi'_1 g_{\mathrm{A}}(\psi'_1)$ of the network nodes, where $g_{\mathrm{A}}(\psi'_1)$ is defined by Eqs. (1.5) and (1.6). Since a fraction q_{B} of nodes from network B depends on nodes from network A, the number of nodes in network B that become nonfunctional is $(1 - \psi_1)q_{\mathrm{B}} = q_{\mathrm{B}}[1 - \psi'_1 g_{\mathrm{A}}(\psi'_1)]$. Accordingly, the remaining fraction of network B nodes is $\phi'_1 = 1 - q_{\mathrm{B}}[1 - \psi'_1 g_{\mathrm{A}}(\psi'_1)]$, and the fraction of nodes in the giant component of network B is $\phi_1 = \phi'_1 g_{\mathrm{B}}(\phi'_1)$.

Following this approach we construct the sequence, ψ'_t and ϕ'_t, of the remaining fraction of nodes at each stage of the cascade of failures. The general form is given by

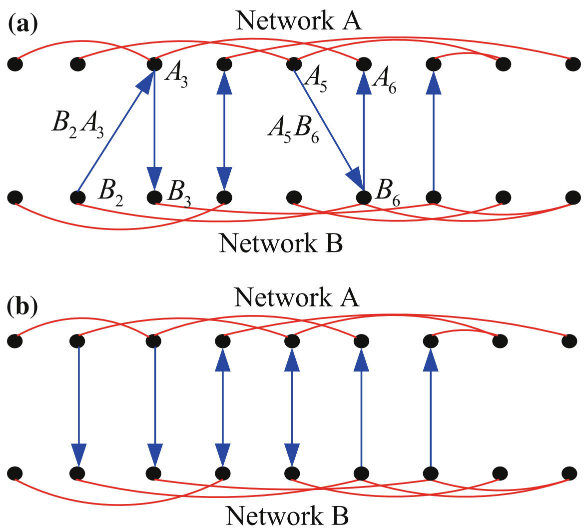

Fig. 1.3 Description of differences between the (**a**) feedback condition and (**b**) no-feedback condition. In the case (**a**), node A_3 depends on node B_2, and node $B_3 \neq B_2$ depends on node A_3, while in case (**b**) this is forbidden. In case (**a**), when $q = 1$ both networks will collapse if one node is removed from one network, which is far from being real. So in our model, we use the no-feedback condition [case (**b**)]. The *blue* links between two networks show the dependency links and the *red* links in each network show the connectivity links which enable each network to function. After [47]

$$\begin{aligned}
&\psi'_1 \equiv p,\\
&\phi'_1 = 1 - q_B[1 - pg_A(\psi'_1)],\\
&\psi'_t = p[1 - q_A(1 - g_B(\phi'_{t-1}))],\\
&\phi'_t = 1 - q_B[1 - pg_A(\psi'_{t-1})].
\end{aligned} \tag{1.19}$$

To determine the state of the system at the end of the cascade process we look at ψ'_τ and ϕ'_τ at the limit of $\tau \to \infty$. This limit must satisfy the equations $\psi'_\tau = \psi'_{\tau+1}$ and $\phi'_\tau = \phi'_{\tau+1}$ since eventually the clusters stop fragmenting and the fractions of randomly removed nodes at step τ and $\tau+1$ are equal. Denoting $\psi'_\tau = x$ and $\phi'_\tau = y$, we arrive at the stationary state to a system of two equations with two unknowns,

$$\begin{aligned}
&x = p\{1 - q_A[1 - g_B(y)]\},\\
&y = 1 - q_B[1 - g_A(x)p].
\end{aligned} \tag{1.20}$$

The giant components of networks A and B at the end of the cascade of failures are, respectively, $P_{\infty,A} = \psi_\infty = xg_A(x)$ and $P_{\infty,B} = \phi_\infty = yg_B(y)$. The numerical results were obtained by iterating system (1.19), where $g_A(\psi'_t)$ and $g_B(\phi'_t)$ are computed using Eqs. (1.9) and (1.10). Figure 1.5 shows excellent agreement between simulations of cascading failures of two partially interdependent networks with $N = 2 \times 10^5$ nodes and the numerical iterations of system (1.19). In the simu-