MASAKI KASHIWARA
PIERRE SCHAPIRA

Volume 332

Grundlehren
der mathematischen
Wissenschaften

A Series of
Comprehensive Studies
in Mathematics

CATEGORIES AND SHEAVES

Springer

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mathematischen Wissenschaften 332

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Categories and Sheaves

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Library of Congress Control Number: 2005930329

Mathematics Subject Classification (2000): 18A, 18E, 18F10, 18F20, 18G

ISSN 0072-7830
ISBN-10 3-540-27949-0 Springer Berlin Heidelberg New York
ISBN-13 978-3-540-27949-5 Springer Berlin Heidelberg New York

Springer is a part of Springer Science+Business Media
springeronline.com

Printed in The Netherlands

Typesetting: by the authors and TechBooks using a Springer LaTeX macro package

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 11304500 41/TechBooks 5 4 3 2 1 0

Preface

The language of Mathematics has changed drastically since the middle of the twentieth century, in particular after Grothendieck's ideas spread from algebraic geometry to many other subjects. As an enrichment for the notions of sets and functions, categories and sheaves are new tools which appear almost everywhere nowadays, sometimes simply in the role of a useful language, but often as the natural approach to a deeper understanding of mathematics.

Category theory, initiated by Eilenberg and Mac Lane in the forties (see [19, 20]), may be seen as part of a wider movement transcending mathematics, of which structuralism in various areas of knowledge is perhaps another facet. Before the advent of categories, people were used to working with a given set endowed with a given structure (a topological space for example) and to studying its properties. The categorical point of view is essentially different. The stress is placed not upon the objects, but on the relations (the morphisms) between objects within the category. The language is natural and allows one to unify various branches of mathematics and to make unexpected links between seemingly different subjects.

Category theory is elementary in the sense that there are few prerequisites to its study, though it may appear forbiddingly abstract to many people. Indeed, the usual course of mathematical education is not conducive to such a conceptual way of thinking. Most mathematicians are used to manipulating spaces and functions, computing integrals and so on, fewer understand the importance of the difference between an equality and an isomorphism or appreciate the beauty and efficiency of diagrams.

Another fundamental idea is that of a sheaf. Sheaves provide a tool for passing from local to global situations and a good deal of mathematics (and physics) revolves around such questions. Sheaves allow us to study objects that exist locally but not globally, such as the holomorphic functions on the Riemann sphere or the orientation on a Möbius strip, and the cohomology of sheaves measures in some sense the obstruction to passing from local to global.

Jean Leray invented sheaves on a topological space in the forties (see [46] and Houzel's historical notes in [38]). Their importance, however, became more evident through the Cartan Seminar and the work of Serre. Subsequently, Serre's work [62] on the local triviality of algebraic fiber bundles led Grothendieck to the realization that the usual notion of a topological space was not appropriate for algebraic geometry (there being an insufficiency of open subsets), and introduced sites, that is, categories endowed with "Grothendieck topologies" and extended sheaf theory to sites.

The development of homological algebra is closely linked to that of category and sheaf theory. Homological algebra is a vast generalization of linear algebra and a key tool in all parts of mathematics dealing with linear phenomena, for example, representations, abelian sheaves, and so forth. Two milestones are the introduction of spectral sequences by Leray (loc. cit.) and the introduction of derived categories by Grothendieck in the sixties.

In this book, we present categories, homological algebra and sheaves in a systematic and exhaustive manner starting from scratch and continuing with full proofs to an exposition of the most recent results in the literature, and sometimes beyond. We also present the main features and key results of related topics that would deserve a whole book for themselves (e.g., tensor categories, triangulated categories, stacks).

Acknowledgments

All along the preparation of the manuscript, we benefited from numerous constructive comments, remarks and suggestions from Andrea D'Agnolo, Pierre Deligne, Stéphane Guillermou, Bernhard Keller, Amnon Neeman, Pietro Polesello, Raphael Rouquier, Valerio Toledano and Ingo Waschkies. David Coyle has assisted us in questions of English idiom and Liliane Beaulieu in historical questions. It is a pleasure to thank all of them here.

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May 2005

Contents

Introduction

The aim of this book is to describe the topics outlined in the preface, categories, homological algebra and sheaves. We also present the main features and key results in related topics which await a similar full-scale treatment such as, for example, tensor categories, triangulated categories, stacks.

The general theory of categories and functors, with emphasis on inductive and projective limits, tensor categories, representable functors, ind-objects and localization is dealt with in Chaps. 1–7.

Homological algebra, including additive, abelian, triangulated and derived categories, is treated in Chaps. 8–15. Chapter 9 provides the tools (using transfinite induction) which will be used later for presenting unbounded derived categories.

Sheaf theory is treated in Chaps. 16–19 in the general framework of Grothendieck topologies. In particular, the results of Chap. 14 are applied to the study of the derived category of the category of sheaves on a ringed site. We also sketch an approach to the more sophisticated subject of stacks (roughly speaking, sheaves with values in the 2-category of categories) and introduce the important notion of twisted sheaves.

Of necessity we have excluded many exciting developments and applications such as n-categories, operads, A_∞-categories, model categories, among others. Without doubt these new areas will soon be intensively treated in the literature, and it is our hope that the present work will provide a basis for their understanding.

We now proceed to a more detailed outline of the contents of the book.

Chapter 1. We begin by defining the basic notions of categories and functors, illustrated with many classical examples. There are some set–theoretical dangers and to avoid contradictions, we work in a given universe. Universes are presented axiomatically, referring to [64] for a more detailed treatment. Among other concepts introduced in this chapter are morphisms of functors, equivalences of categories, representable functors, adjoint functors and so on. We introduce in particular the category $\mathrm{Fct}(I, \mathcal{C})$ of functors from a small category I to a category $\mathcal{C}$ in a universe $\mathcal{U}$, and look briefly at the 2-category $\mathcal{U}$-**Cat** of all $\mathcal{U}$-categories.

Here, the key result is the Yoneda lemma showing that a category $\mathcal{C}$ may be embedded in the category $\mathcal{C}^\wedge$ of all contravariant functors from $\mathcal{C}$ to **Set**, the category of sets. This allows us in a sense to reduce category theory to set theory and leads naturally to the notion of a representable functor. The category $\mathcal{C}^\wedge$ enjoys most of the properties of the category **Set**, and it is often extremely convenient, if not necessary, to replace $\mathcal{C}$ by $\mathcal{C}^\wedge$, just as in analysis, we are lead to replace functions by generalized functions.

Chapters 2 and 3. Inductive and projective limits are the most important concepts dealt with in this book. They can be seen as the essential tool of category theory, corresponding approximately to the notions of union and intersection in set theory. Since students often find them difficult to master, we provide many detailed examples. The category **Set** is not equivalent to its opposite category, and projective and inductive limits in **Set** behave very differently. Note that inductive and projective limits in a category are both defined as representable functors of *projective* limits in the category **Set**.

Having reached this point we need to construct the *Kan extension* of functors. Consider three categories $J, I, \mathcal{C}$ and a functor $\varphi\colon J \to I$. The functor φ defines by composition a functor φ_* from $\mathrm{Fct}(I, \mathcal{C})$ to $\mathrm{Fct}(J, \mathcal{C})$, and we can construct a right or left adjoint for this functor by using projective or inductive limits. These constructions will systematically be used in our presentation of sheaf theory and correspond to the operations of direct or inverse images of sheaves.

Next, we cover two essential tools for the study of limits in detail: cofinal functors (roughly analogous to the notion of extracted sequences in analysis) and filtrant[1] categories (which generalizes the notion of a directed set). As we shall see in this book, filtrant categories are of fundamental importance.

We define right exact functors (and similarly by reversing the arrows, left exact functors). Given that finite inductive limits exist, a functor is right exact if and only if it commutes with such limits.

Special attention is given to the category **Set** and to the study of filtrant inductive limits in **Set**. We prove in particular that inductive limits in **Set** indexed by a small category I commute with finite projective limits if and only if I is filtrant.

Chapter 4. Tensor categories axiomatize the properties of tensor products of vector spaces. Nowadays, tensor categories appear in many areas, mathematical physics, knot theory, computer science among others. They acquired popular attention when it was found that quantum groups produce rich examples of non-commutative tensor categories. Tensor categories and their applications in themselves merit an extended treatment, but we content ourselves

[1] Some authors use the terms "filtered" or "filtering". We have chosen to keep the French word.

here with a rapid treatment referring the reader to [15, 40] and [59] from the vast literature on this subject.

Chapter 5. We give various criteria for a functor with values in **Set** to be representable and, as a by-product, obtain criteria under which a functor will have an adjoint. This necessitates the introduction of two important notions: strict morphisms and systems of generators (and in particular, a generator) in a category $\mathcal{C}$. References are made to [64].

Chapter 6. The Yoneda functor, which sends a category $\mathcal{C}$ to $\mathcal{C}^\wedge$, enjoys many pleasing properties, such as that of being fully faithful and commuting with projective limits, but it is not right exact.

The category $\mathrm{Ind}(\mathcal{C})$ of ind-objects of $\mathcal{C}$ is the subcategory of $\mathcal{C}^\wedge$ consisting of small and filtrant inductive limits of objects in $\mathcal{C}$. This category has many remarkable properties: it contains $\mathcal{C}$ as a full subcategory, admits small filtrant inductive limits, and the functor from $\mathcal{C}$ to $\mathrm{Ind}(\mathcal{C})$ induced by the Yoneda functor is now right exact. On the other hand, we shall show in Chap. 15 that in the abelian case, $\mathrm{Ind}(\mathcal{C})$ does not in general have enough injective objects when we remain in a given universe.

This theory, introduced in [64] (see also [3] for complementary material) was not commonly used until recently, even by algebraic geometers, but matters are rapidly changing and ind-objects are increasingly playing an important role.

Chapter 7. The process of localization appears everywhere and in many forms in mathematics. Although natural, the construction is not easy in a categorical setting. As usual, it is easier to embed than to form quotient.

If a category $\mathcal{C}$ is localized with respect to a family of morphisms $\mathcal{S}$, the morphisms of $\mathcal{S}$ become isomorphisms in the localized category $\mathcal{C}_\mathcal{S}$ and if $F\colon \mathcal{C} \to \mathcal{A}$ is a functor which sends the morphisms in $\mathcal{S}$ to isomorphisms in $\mathcal{A}$, then F will factor uniquely through the natural functor $Q\colon \mathcal{C} \to \mathcal{C}_\mathcal{S}$. This is the aim of localization. We construct the localization of $\mathcal{C}$ when $\mathcal{S}$ satisfies suitable conditions, namely, when $\mathcal{S}$ is a (right or left) multiplicative system.

Interesting features appear when we try to localize a functor F that is defined on $\mathcal{C}$ with values in some category $\mathcal{A}$, and does not map the arrows in $\mathcal{S}$ to isomorphisms in $\mathcal{A}$. Even in this case, we can define the right or left localization of the functor F under suitable conditions. We interpret the right localization functor as a left adjoint to the composition with the functor Q, and this adjoint exists if $\mathcal{A}$ admits inductive limits. It is then a natural idea to replace the category $\mathcal{A}$ with that of ind-objects of $\mathcal{A}$, and check whether the localization of F at $X \in \mathcal{C}$ is representable in $\mathcal{A}$. This is the approach taken by Deligne [17] which we follow here.

Localization is an essential step in constructing derived categories. A classical reference for localization is [24].

Chapter 8. The standard example of abelian categories is the category $\mathrm{Mod}(R)$ of modules over a ring R. Additive categories present a much weaker structure which appears for example when considering special classes of modules (e.g. the category of projective modules over the ring R is additive but not abelian).

The concept of abelian categories emerged in the early 1950s (see [13]). They inherit all the main properties of the category $\mathrm{Mod}(R)$ and form a natural framework for the development of homological algebra, as is shown in the subsequent chapters. Of particular importance are the Grothendieck categories, that is, abelian categories which admit (exact) small filtrant inductive limits and a generator. We prove in particular the Gabriel-Popescu theorem (see [54]) which asserts that a Grothendieck category may be embedded into the category of modules over the ring of endomorphisms of a generator.

We also study the abelian category $\mathrm{Ind}(\mathcal{C})$ of ind-objects of an abelian category $\mathcal{C}$ and show in particular that the category $\mathrm{Ind}(\mathcal{C})$ is abelian and that the natural functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ is exact. Finally we prove that under suitable hypotheses, the Kan extension of a right (or left) exact functor defined on an additive subcategory of an abelian category is also exact. Classical references are the book [14] by Cartan-Eilenberg, and Grothendieck's paper [28] which stresses the role of abelian categories, derived functors and injective objects.

An important source of historical information on this period is given in [16] by two of the main contributors.

Chapter 9. In this chapter we extend many results on filtrant inductive limits to the case of π-filtrant inductive limits, for an infinite cardinal π. An object X is π-accessible if $\mathrm{Hom}_{\mathcal{C}}(X, \cdot)$ commutes with π-filtrant inductive limits. We specify conditions which ensure that the category $\mathcal{C}_\pi$ of π-accessible objects is small and that the category of its ind-objects is equivalent $\mathcal{C}$. These techniques are used to prove that, under suitable hypotheses, given a family $\mathcal{F}$ of morphisms in a category $\mathcal{C}$, there are *enough $\mathcal{F}$-injective* objects.

Some arguments developed here were initiated in Grothendieck's paper [28] and play an essential role in the theory of model categories (see [56] and [32]). They are used in Chap. 14 in proving that the derived category of a Grothendieck category admits enough homotopically injective objects.

Here, we give two important applications. The first one is the fact that a Grothendieck category possesses enough injective objects. The second one is the Freyd-Mitchell theorem which asserts that any small abelian category may be embedded in the category of modules over a suitable ring. References are made to [64]. Accessible objects are also discussed in [1, 23] and [49].

Chapter 10. Triangulated categories first appeared implicitly in papers on stable homotopy theory after the work of Puppe [55], until Verdier axiomatized the properties of these categories (we refer to the preface by L. Illusie of [69] for more historical comments). Triangulated categories are now very popular and are part of the basic language in various branches of mathematics, especially algebraic geometry (see e.g. [57, 70]), algebraic topology and representation theory (see e.g. [35]). They appeared in analysis in the early 1970s under the influence of Mikio Sato (see [58]) and more recently in symplectic geometry after Kontsevich expressed mirror symmetry (see [43]) using this language.

A category endowed with an automorphism T is called here a category with translation. In such a category, a triangle is a sequence of morphisms $X \to Y \to Z \to T(X)$. A triangulated category is an additive category with translation endowed with a family of so-called *distinguished triangles* satisfying certain axioms. Although the first example of a triangulated category only appears in the next chapter, it seems worthwhile to develop this very elegant and easy formalism here for its own sake.

In this chapter, we study the localization of triangulated categories and the construction of cohomological functors in some detail. We also give a short proof of the Brown representability theorem [11], in the form due to Neeman [53], which asserts that, under suitable hypotheses, a contravariant cohomological functor defined on a triangulated category which sends small direct sums to products is representable.

We do not treat t-structures here, referring to the original paper [4] (see [38] for an expository treatment).

Chapter 11. It is perhaps the main idea of homological algebra to replace an object in a category $\mathcal{C}$ by a complex of objects of $\mathcal{C}$, the components of which have "good properties". For example, when considering the tensor product and its derived functors, we replace a module by a complex of projective (or flat) modules and, when considering the global-section functor and its derived functors, we replace a sheaf by a complex of flabby sheaves.

It is therefore natural to study the category $\mathrm{C}(\mathcal{C})$ of complexes of objects of an additive category $\mathcal{C}$. This category inherits an automorphism, the *shift functor*, called the "suspension" by algebraic topologists. Other basic constructions borrowed from algebraic topology are that of the *mapping cone* of a morphism and that of *homotopy* of complexes. In fact, in order to be able to work, i.e., to form commutative diagrams, we have to make morphisms in $\mathrm{C}(\mathcal{C})$ which are homotopic to zero, actually isomorphic to zero. This defines the homotopy category $\mathrm{K}(\mathcal{C})$ and the main result (stated in the slightly more general framework of additive categories with translation) is that $\mathrm{K}(\mathcal{C})$ is triangulated.

Many complexes, such as Čech complexes in sheaf theory (see Chap. 18 below), are obtained naturally by simplicial construction. Here, we construct complexes associated with simplicial objects and give a criterion for these complexes to be homotopic to zero.

When considering bifunctors on additive categories, we are rapidly lead to consider the category $\mathrm{C}(\mathrm{C}(\mathcal{C}))$ of complexes of complexes (i.e., double complexes), and so on. We explain here how a diagonal procedure allows us, under suitable hypotheses, to reduce a double complex to a simple one. Delicate questions of signs arise and necessitate careful treatment.

Chapter 12. When $\mathcal{C}$ is abelian, we can define the j-th cohomology object $H^j(X)$ of a complex X. The main result is that the functor H^j is cohomological, that is, sends distinguished triangles in $\mathrm{K}(\mathcal{C})$ to long exact sequences in $\mathcal{C}$.

When a functor F with values in $\mathcal{C}$ is defined on the category of finite sets, it is possible to attach to F a complex in $\mathcal{C}$, generalizing the classical notion of Koszul complexes. We provide the tools needed to calculate the cohomology of such complexes and treat some examples such as distributive families of subobjects.

We also study the cohomology of a double complex, replacing the Leray's traditional spectral sequences by an intensive use of the truncation functors. We find this approach much easier and perfectly adequate in practice.

Chapter 13. Constructing the derived category of an abelian category is easy with the tools now at hand. It is nothing more than the localization of the homotopy category $\mathrm{K}(\mathcal{C})$ with respect to exact complexes.

Here we give the main constructions and results concerning derived categories and functors, including some new results.

Despite their popularity, derived categories are sometimes supposed difficult. A possible reason for this reputation is that to date there has been no systematic, pedagogical treatment of the theory. The classical texts on derived categories are the famous Hartshorne Notes [31], or Verdier's résumé of his thesis [68] (of which the complete manuscript has been published recently [69]). Apart from these, there are a few others which may be found in particular in the books [25, 38] and [71]. Recall that the original idea of derived categories goes back to Grothendieck.

Chapter 14. Using the results of Chap. 9, we study the (unbounded) derived category $\mathrm{D}(\mathcal{C})$ of a Grothendieck category $\mathcal{C}$. First, we show that any complex in a Grothendieck category is quasi-isomorphic to a *homotopically injective complex* and we deduce the existence of right derived functors in $\mathrm{D}(\mathcal{C})$. We then prove that the Brown representability theorem holds in $\mathrm{D}(\mathcal{C})$ and discuss the existence of left derived functors, as well as the composition of (right or left) derived functors and derived adjunction formulas.

Spaltenstein [65] was the first to consider unbounded complexes and the corresponding derived functors. The (difficult) result which asserts that the Brown representability theorem holds in the derived category of a Grothendieck category seems to be due to independently to [2] and [21] (see also [6, 42, 53] and [44]). Note that most of the ideas presented here come from topology, in which context the names of Adams, Bousfield, Kan, Thomason among others should be mentioned.

Chapter 15. We study here the derived category of the category $\mathrm{Ind}(\mathcal{C})$ of ind-objects of an abelian category $\mathcal{C}$. Things are not easy since in the simple case where $\mathcal{C}$ is the category of vector spaces over a field k, the category $\mathrm{Ind}(\mathcal{C})$ does not have enough injective objects. In order to overcome this difficulty, we introduce the notion of quasi-injective objects. We show that under suitable hypotheses, there are enough such objects and that they allow us to derive functors. We also study some links between the derived category of $\mathrm{Ind}(\mathcal{C})$ and that of ind-objects of the derived category of $\mathcal{C}$. Note that the category of ind-objects of a triangulated category does not seem to be triangulated.

Most of the results in this chapter are new and we hope that they may be useful. They are so when applied to the construction of ind-sheaves, for which we refer to [39].

Chapter 16. The notion of sheaves relies on that of coverings and a Grothendieck topology on a category is defined by axiomatizing the notion of coverings.

In this chapter we give the axioms for Grothendieck topologies using sieves and then introduce the notions of local epimorphisms and local isomorphisms. We give several examples and study the properties of the family of local isomorphisms in detail, showing in particular that this family is stable under inductive limits. The classical reference is [64].

Chapter 17. A site X is a category $\mathcal{C}_X$ endowed with a Grothendieck topology. A presheaf F on X with values in a category $\mathcal{A}$ is a contravariant functor on $\mathcal{C}_X$ with values in $\mathcal{A}$, and a presheaf F is a sheaf if, for any local isomorphism $A \to U$, $F(U) \to F(A)$ is an isomorphism. When $\mathcal{C}_X$ is the category of open subsets of a topological space X, we recover a familiar notion.

Here, we construct the sheaf F^a associated with a presheaf F with values in a category $\mathcal{A}$ satisfying suitable properties. We also study restriction and extension of sheaves, direct and inverse images, and internal $\mathcal{H}om$. However, we do not enter the theory of Topos, referring to [64] (see also [48] for further exciting developments).

Chapter 18. When $\mathcal{O}_X$ is a sheaf of rings on a site X, we define the category $\mathrm{Mod}(\mathcal{O}_X)$ of sheaves of $\mathcal{O}_X$-modules. This is a Grothendieck category to which we may apply the tools obtained in Chap. 14.

In this Chapter, we construct the unbounded derived functors $R\mathcal{H}om_{\mathcal{O}_X}$ of internal hom, $\overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X}$ of tensor product, Rf_* of direct image and Lf^* of inverse image (these two last functors being associated with a morphism f of ringed sites) and we study their relations. Such constructions are well-known in the case of bounded derived categories, but the unbounded case, initiated by Spaltenstein [65], is more delicate.

We do not treat proper direct images and duality for sheaves. Indeed, there is no such theory for sheaves on abstract sites, where the construction in the algebraic case for which we refer to [17], differs from that in the topological case for which we refer to [38].

Chapter 19. The notion of constant functions is not local and it is more natural (and useful) to consider locally constant functions. The presheaf of such functions is in fact a sheaf, called a constant sheaf. There are however sheaves which are locally, but not globally, isomorphic to this constant sheaf, and this leads us to the fundamental notion of locally constant sheaves, or local systems. The orientation sheaf on a real manifold is a good example of such a sheaf. We consider similarly categories which are locally equivalent to the category of sheaves, which leads us to the notions of stacks and twisted sheaves.

A stack on a site X is, roughly speaking, a sheaf of categories, or, more precisely, a sheaf with values in the 2-category of all $\mathcal{U}$-categories of a given universe $\mathcal{U}$. Indeed, it would be possible to consider higher objects (n-stacks), but we do not pursue this matter here. This new field of mathematics was first explored in the sixties by Grothendieck and Giraud (see [26]) and after having been long considered highly esoteric, it is now the object of intense activity from algebraic geometry to theoretical physics. Note that 2-categories were first introduced by Bénabou (see [5]), a student of an independent-minded category theorist, Charles Ehresmann.

This last chapter should be understood as a short presentation of possible directions in which the theory may develop.

1

The Language of Categories

A set E is a collection of elements, and given two elements x and y in E there are no relations between x and y. The notion of a category is more sophisticated. A category $\mathcal{C}$ possesses objects similarly as a set possesses elements, but now for each pair of objects X and Y in $\mathcal{C}$, one is given a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ called the morphisms from X to Y, representing possible relations between X and Y.

Once we have the notion of a category, it is natural to ask what are the morphisms from a category to another, and this lead to the notion of functors. We can also define the morphisms of functors, and as a byproduct, the notion of an equivalence of categories. At this stage, it would be tempting to define the notion of a 2-category, but this is out of the scope of this book.

The cornerstone of Category Theory is the Yoneda lemma. It asserts that a category $\mathcal{C}$ may be embedded in the category $\mathcal{C}^\wedge$ of all contravariant functors from this category to the category **Set** of sets, the morphisms in **Set** being the usual maps. This allows us, in some sense, to reduce Category Theory to Set Theory. The Yoneda lemma naturally leads to the notion of representable functor, and in particular to that of adjoint functor.

To a category $\mathcal{C}$, we can associate its opposite category $\mathcal{C}^{\mathrm{op}}$ obtained by reversing the arrows, and in this theory most of the constructions have their counterparts, monomorphism and epimorphism, right adjoint and left adjoint, etc. Of course, when a statement may be deduced from another one by reversing the arrows, we shall simply give one of the two statements. But the category **Set** is not equivalent to its opposite category, and **Set** plays a very special role in the whole theory. For example, inductive and projective limits in categories are constructed by using projective limits in **Set**.

A first example of a category would be the category **Set** mentioned above. But at this stage one encounters a serious difficulty, namely that of manipulating “all” sets. Moreover, we constantly use the category of all functors from a given category to **Set**. In this book, to avoid contradictions, we work in a given universe. Here, we shall begin by briefly recalling the axioms of universes, referring to [64] for more details.

1.1 Preliminaries: Sets and Universes

The aim of this section is to fix some notations and to recall the axioms of universes. We do not intend neither to enter Set Theory, nor to say more about universes than what we need. For this last subject, references are made to [64].

For a set u, we denote as usual by $\mathcal{P}(u)$ the set of subsets of $u\colon \mathcal{P}(u) = \{x; x \subset u\}$. For $x_1, \ldots x_n$, we denote as usual by $\{x_1, \ldots, x_n\}$ the set whose elements are $x_1, \ldots x_n$.

Definition 1.1.1. *A* universe $\mathcal{U}$ *is a set satisfying the following properties:*

(i) $\emptyset \in \mathcal{U}$,
(ii) $u \in \mathcal{U}$ *implies* $u \subset \mathcal{U}$, (*equivalently,* $x \in \mathcal{U}$ *and* $y \in x$ *implies* $y \in \mathcal{U}$, *or else* $\mathcal{U} \subset \mathcal{P}(\mathcal{U})$),
(iii) $u \in \mathcal{U}$ *implies* $\{u\} \in \mathcal{U}$,
(iv) $u \in \mathcal{U}$ *implies* $\mathcal{P}(u) \in \mathcal{U}$,
(v) *if* $I \in \mathcal{U}$ *and* $u_i \in \mathcal{U}$ *for all* $i \in I$, *then* $\bigcup_{i \in I} u_i \in \mathcal{U}$,
(vi) $\mathbb{N} \in \mathcal{U}$.

As a consequence we have

(vii) $u \in \mathcal{U}$ implies $\bigcup_{x \in u} x \in \mathcal{U}$,
(viii) $u, v \in \mathcal{U}$ implies $u \times v \in \mathcal{U}$,
(ix) $u \subset v \in \mathcal{U}$ implies $u \in \mathcal{U}$,
(x) if $I \in \mathcal{U}$ and $u_i \in \mathcal{U}$ for all $i \in I$, then $\prod_{i \in I} u_i \in \mathcal{U}$.

Following Grothendieck, we shall add an axiom to the Zermelo-Fraenkel theory, asking that for any set X there exists a universe $\mathcal{U}$ such that $X \in \mathcal{U}$. For more explanations, refer to [64].

Definition 1.1.2. *Let* $\mathcal{U}$ *be a universe.*

(i) *A set is called a* $\mathcal{U}$-set *if it belongs to* $\mathcal{U}$.
(ii) *A set is called* $\mathcal{U}$-small *if it is isomorphic to a set belonging to* $\mathcal{U}$.

Definition 1.1.3. (i) *An* order *on a set* I *is a relation* $\leq$ *which is:*
  (a) *reflexive, that is,* $i \leq i$ *for all* $i \in I$,
  (b) *transitive, that is,* $i \leq j$, $j \leq k \Rightarrow i \leq k$,
  (c) *anti-symmetric, that is,* $i \leq j$, $j \leq i \Rightarrow i = j$.

(ii) *An order is* directed (*we shall also say* "filtrant") *if* I *is non empty and if for any* $i, j \in I$, *there exists* $k \in I$ *such that* $i \leq k$ *and* $j \leq k$.
(iii) *An order is* total (*some authors say "linear"*) *if for any* $i, j \in I$, *one has* $i \leq j$ *or* $j \leq i$.
(iv) *An ordered set* I *is* inductively ordered *if any totally ordered subset* J *of* I *has an upper bound* (*i.e., there exists* $a \in I$ *such that* $j \leq a$ *for all* $j \in J$).
(v) *If* $\leq$ *is an order on* I, $<$ *is the relation given by* $x < y$ *if and only if* $x \leq y$ *and* $x \neq y$. *We also write* $x \geq y$ *if* $y \leq x$ *and* $x > y$ *if* $y < x$.

Recall that Zorn's lemma asserts that any inductively ordered set admits a maximal element.

Notations 1.1.4. (i) We denote by $\{\mathrm{pt}\}$ a set with one element, and this single element is often denoted by pt. We denote by $\emptyset$ the set with no element.
(ii) In all this book, a ring means an associative ring with unit, and the action of a ring on a module is unital. If there is no risk of confusion, we simply denote by 0 the module with a single element. A field is a non-zero commutative ring in which every non-zero element is invertible.
(iii) We shall often denote by k a commutative ring. A k-algebra is a ring R endowed with a morphism of rings $\varphi\colon k \to R$ such that the image of k is contained in the center of R. We denote by $k^\times$ the group of invertible elements of k.
(iv) As usual, we denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{Q}$ (resp. $\mathbb{R}$, resp. $\mathbb{C}$) the field of rational numbers (resp. real numbers, resp. complex numbers). We denote by $\mathbb{N}$ the set of non-negative integers, that is, $\mathbb{N} = \{n \in \mathbb{Z}\,;\, n \geq 0\}$.
(v) We denote by $k[x_1, \dots, x_n]$ the ring of polynomials in the variables $x_1, \dots, x_n$ over a commutative ring k.
(vi) We denote by δ_{ij} the Kronecker symbol, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

1.2 Categories and Functors

Definition 1.2.1. *A* category $\mathcal{C}$ *consists of:*

(i) *a set* $\mathrm{Ob}(\mathcal{C})$*,*
(ii) *for any* $X, Y \in \mathrm{Ob}(\mathcal{C})$*, a set* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$*,*
(iii) *for any* $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$*, a map:*

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

called the composition and denoted by $(f, g) \mapsto g \circ f$*,*

these data satisfying:

(a) $\circ$ *is associative, i.e., for* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$*,* $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ *and* $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$*, we have* $(h \circ g) \circ f = h \circ (g \circ f)$*,*
(b) *for each* $X \in \mathrm{Ob}(\mathcal{C})$*, there exists* $\mathrm{id}_X \in \mathrm{Hom}\,(X, X)$ *such that* $f \circ \mathrm{id}_X = f$ *for all* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ *and* $\mathrm{id}_X \circ g = g$ *for all* $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$*.*

An element of $\mathrm{Ob}(\mathcal{C})$ is called an *object* of $\mathcal{C}$ and for $X, Y \in \mathrm{Ob}(\mathcal{C})$, an element of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is called a *morphism* (from X to Y) in $\mathcal{C}$. The morphism id_X is called the *identity morphism* (or the identity, for short) of X. Note that there is a unique $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ satisfying the condition in (b).

A category $\mathcal{C}$ is called a $\mathcal{U}$-category if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is $\mathcal{U}$-small for any $X, Y \in \mathrm{Ob}(\mathcal{C})$.

A $\mathcal{U}$-small category is a $\mathcal{U}$-category $\mathcal{C}$ such that $\mathrm{Ob}(\mathcal{C})$ is $\mathcal{U}$-small.

Notation 1.2.2. We often write $X \in \mathcal{C}$ instead of $X \in \mathrm{Ob}(\mathcal{C})$, and $f\colon X \to Y$ or else $f\colon Y \leftarrow X$ instead of $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. We say that X is the *source* and Y the *target* of f. We sometimes call f an *arrow* instead of "a morphism".

We introduce the *opposite category* $\mathcal{C}^{\mathrm{op}}$ by setting:

$$\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C}), \quad \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X),$$

and defining the new composition $g \overset{\mathrm{op}}{\circ} f$ of $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y, Z)$ by $g \overset{\mathrm{op}}{\circ} f = f \circ g$. For an object X or a morphism f in $\mathcal{C}$, we shall sometimes denote by X^{op} or f^{op} its image in $\mathcal{C}^{\mathrm{op}}$. In the sequel, we shall simply write $\circ$ instead of $\overset{\mathrm{op}}{\circ}$.

A morphism $f\colon X \to Y$ is an *isomorphism* if there exists $g\colon X \leftarrow Y$ such that $f \circ g = \mathrm{id}_Y$, $g \circ f = \mathrm{id}_X$. Such a g, which is unique, is called the *inverse* of f and is denoted by f^{-1}. If $f\colon X \to Y$ is an isomorphism, we write $f\colon X \xrightarrow{\sim} Y$. If there is an isomorphism $X \xrightarrow{\sim} Y$, we say that X and Y are isomorphic and we write $X \simeq Y$.

An *endomorphism* is a morphism with same source and target, that is, a morphism $f\colon X \to X$.

An *automorphism* is an endomorphism which is an isomorphism.

Two morphisms f and g are *parallel* if they have same source and same target, visualized by $f, g\colon X \rightrightarrows Y$.

A morphism $f\colon X \to Y$ is a *monomorphism* if for any pair of parallel morphisms $g_1, g_2\colon Z \rightrightarrows X$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

A morphism $f\colon X \to Y$ is an *epimorphism* if $f^{\mathrm{op}}\colon Y^{\mathrm{op}} \to X^{\mathrm{op}}$ is a monomorphism in $\mathcal{C}^{\mathrm{op}}$. Hence, f is an epimorphism if and only if for any pair of parallel morphisms $g_1, g_2\colon Y \rightrightarrows Z$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

Note that f is a monomorphism if and only if the map $f\circ\colon \mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ is injective for any object Z, and f is an epimorphism if and only if the map $\circ f\colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is injective for any object Z.

Also note that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms and if f and g are monomorphisms (resp. epimorphisms, resp. isomorphisms), then $g \circ f$ is a monomorphism (resp. epimorphism, resp. isomorphism).

We sometimes write $f\colon X \rightarrowtail Y$ or else $f\colon X \hookrightarrow Y$ to denote a monomorphism and $f\colon X \twoheadrightarrow Y$ to denote an epimorphism.

For two morphisms $f\colon X \to Y$ and $g\colon Y \to X$ satisfying $f \circ g = \mathrm{id}_Y$, f is called a *left inverse* of g and g is called a *right inverse* of f. We also say that g is a *section* of f or f is a *cosection* of g. In such a situation, f is an epimorphism and g a monomorphism.

A category $\mathcal{C}'$ is a *subcategory* of $\mathcal{C}$, denoted by $\mathcal{C}' \subset \mathcal{C}$, if: $\mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for any $X, Y \in \mathcal{C}'$, the composition in $\mathcal{C}'$ is induced by the composition in $\mathcal{C}$ and the identity morphisms in $\mathcal{C}'$ are identity morphisms in $\mathcal{C}$. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is *full* if $\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathcal{C}'$. A full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is *saturated* if $X \in \mathcal{C}$ belongs to $\mathcal{C}'$ whenever X is isomorphic to an object of $\mathcal{C}'$.

A category is *discrete* if all the morphisms are the identity morphisms.

A category $\mathcal{C}$ is *non empty* if $\mathrm{Ob}(\mathcal{C})$ is non empty.

A category $\mathcal{C}$ is a *groupoid* if all morphisms are isomorphisms.

A category $\mathcal{C}$ is *finite* if the set of all morphisms in $\mathcal{C}$ (hence, in particular, the set of objects) is a finite set.

A category $\mathcal{C}$ is *connected* if it is non empty and for any pair of objects $X, Y \in \mathcal{C}$, there is a finite sequence of objects $(X_0, \dots, X_n)$, $X_0 = X$, $X_n = Y$, such that at least one of the sets $\mathrm{Hom}_{\mathcal{C}}(X_j, X_{j+1})$ or $\mathrm{Hom}_{\mathcal{C}}(X_{j+1}, X_j)$ is non empty for any $j \in \mathbb{N}$ with $0 \le j \le n-1$.

Remark that a *monoid* M (i.e., a set endowed with an internal product with associative and unital law) is nothing but a category with only one object. (To M, associate the category $\mathcal{M}$ with the single object a and morphisms $\mathrm{Hom}_{\mathcal{M}}(a, a) = M$.) Similarly, a group G defines a groupoid, namely the category $\mathcal{G}$ with a single object a and morphisms $\mathrm{Hom}_{\mathcal{G}}(a, a) = G$.

A *diagram* in a category $\mathcal{C}$ is a family of symbols representing objects of $\mathcal{C}$ and a family of arrows between these symbols representing morphisms of these objects. One defines in an obvious way the notion of a *commutative diagram*. For example, consider the diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "g"] \\ V \arrow[r, "k"] & Z, \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "l"] & Y \arrow[r, "g"] & Z\,. \end{tikzcd}$$

Then the first diagram is commutative if and only if $g \circ f = k \circ h$ and the second diagram is commutative if and only if $g \circ f = l$.

Notation 1.2.3. We shall also encounter diagrams such as:

$$\begin{tikzcd} Z \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & X \arrow[r, "f"] & Y \end{tikzcd}\quad . \tag{1.2.1}$$

We shall say that the two compositions coincide if $f \circ g_1 = f \circ g_2$.

We shall also encounter diagrams of categories. (See Remark 1.3.6 below.)

Examples 1.2.4. (i) **Set** is the category of $\mathcal{U}$-sets and maps, $\mathbf{Set}^f$ the full subcategory consisting of finite $\mathcal{U}$-sets. If we need to emphasize the universe $\mathcal{U}$, we write $\mathcal{U}$-**Set** instead of **Set**. Note that the category of all sets is not a category since the collection of all sets is not a set. This is one of the reasons why we have to introduce a universe $\mathcal{U}$.
(ii) The category **Rel** of binary relations is defined by: $\mathrm{Ob}(\mathbf{Rel}) = \mathrm{Ob}(\mathbf{Set})$ and $\mathrm{Hom}_{\mathbf{Rel}}(X, Y) = \mathcal{P}(X \times Y)$, the set of subsets of $X \times Y$. The composition law is defined as follows. If $f\colon X \to Y$ and $g\colon Y \to Z$, $g \circ f$ is the set

$$\{(x, z) \in X \times Z; \text{ there exists } y \text{ such that } (x, y) \in f, (y, z) \in g\}.$$

Of course, id_X is the diagonal set of $X \times X$.

Notice that **Set** is a subcategory of **Rel**, but is not a full subcategory.

(iii) **pSet** is the category of pointed $\mathcal{U}$-sets. An object of **pSet** is a pair (X, x) with a $\mathcal{U}$-set X and $x \in X$. A morphism $f : (X, x) \to (Y, y)$ is a map $f : X \to Y$ such that $f(x) = y$.

(iv) Let R be a ring (with $R \in \mathcal{U}$). The category of left R-modules belonging to $\mathcal{U}$ and R-linear maps is denoted $\mathrm{Mod}(R)$. Hence, by definition, $\mathrm{Hom}_{\mathrm{Mod}(R)}(\cdot\,,\cdot) = \mathrm{Hom}_R(\cdot\,,\cdot)$. Recall that right R-modules are left R^{op}-modules, where R^{op} denotes the ring R with the opposite multiplicative structure. Note that $\mathrm{Mod}(\mathbb{Z})$ is the category of abelian groups.

We denote by $\mathrm{End}_R(M)$ the ring of R-linear endomorphisms of an R-module M and by $\mathrm{Aut}_R(M)$ the group of R-linear automorphisms of M.

We denote by $\mathrm{Mod}^{\mathrm{f}}(R)$ the full subcategory of $\mathrm{Mod}(R)$ consisting of *finitely generated* R-modules. (Recall that M is finitely generated if there exists a surjective R-linear map $u : R^{\oplus n} \twoheadrightarrow M$ for some integer $n \geq 0$.) One also says *of finite type* instead of "finitely generated".

We denote by $\mathrm{Mod}^{\mathrm{fp}}(R)$ the full subcategory of $\mathrm{Mod}^{\mathrm{f}}(R)$ consisting of R-modules of *finite presentation.* (Recall that M is of finite presentation if it is of finite type and moreover the kernel of the linear map u above is of finite type.)

(v) Let $(I, \leq)$ be an ordered set. We associate to it a category $\mathcal{I}$ as follows.

$$\mathrm{Ob}(\mathcal{I}) = I$$
$$\mathrm{Hom}_{\mathcal{I}}(i, j) = \begin{cases} \{\mathrm{pt}\} & \text{if } i \leq j\,, \\ \emptyset & \text{otherwise}\,. \end{cases}$$

In other words, the set of morphisms from i to j has a single element if $i \leq j$, and is empty otherwise. Note that $\mathcal{I}^{\mathrm{op}}$ is the category associated to $(I, \leq^{\mathrm{op}})$, where $x \leq^{\mathrm{op}} y$ if and only if $y \leq x$. In the sequel, we shall often simply write I instead of $\mathcal{I}$. (See Exercise 1.3 for a converse construction.)

(vi) We denote by **Top** the category of topological spaces belonging to $\mathcal{U}$ and continuous maps.

The set of all morphisms of a category $\mathcal{C}$ may be endowed with a structure of a category.

Definition 1.2.5. *Let $\mathcal{C}$ be a category. We denote by* $\mathrm{Mor}(\mathcal{C})$ *the category whose objects are the morphisms in $\mathcal{C}$ and whose morphisms are described as follows. Let $f : X \to Y$ and $g : X' \to Y'$ belong to* $\mathrm{Mor}(\mathcal{C})$*. Then* $\mathrm{Hom}_{\mathrm{Mor}(\mathcal{C})}(f, g) = \{u : X \to X', v : Y \to Y'; g \circ u = v \circ f\}$*. The composition and the identity in* $\mathrm{Mor}(\mathcal{C})$ *are the obvious ones.*