ANSWER BOOK FOR

# CALCULUS

Third Edition

*Michael Spivak*

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**1.** **(ii)**

$$\begin{aligned}(x-y)(x+y) &= [x+(-y)](x+y) = x(x+y)+(-y)(x+y)\\ &= x(x+y) - [y(x+y)] = x^2+xy-[yx+y^2]\\ &= x^2+xy-y^2 = x^2-y^2.\end{aligned}$$

**(iv)**

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= x(x^2+xy+y^2) - [y(x^2+xy+y^2)]\\ &= x^3+x^2y+xy^2-[yx^2+xy^2+y^3] = x^3-y^3.\end{aligned}$$

**(v)**

$$\begin{aligned}&(x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})\\ &\qquad = x(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})\\ &\qquad\qquad - [y(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})]\\ &\qquad = x^n+x^{n-1}y+\cdots+x^2y^{n-2}+xy^{n-1}\\ &\qquad\qquad - [x^{n-1}y+x^{n-2}y^2+\cdots+xy^{n-1}+y^n]\\ &\qquad = x^n-y^n.\end{aligned}$$

Using the notation of Chapter 2, this proof can be written as follows:

$$\begin{aligned}(x-y)\cdot\sum_{j=0}^{n-1}x^jy^{n-1-j} &= x\left(\sum_{j=0}^{n-1}x^jy^{n-1-j}\right) - \left[y\left(\sum_{j=0}^{n-1}x^jy^{n-1-j}\right)\right]\\ &= x^n+\sum_{j=0}^{n-2}x^{j+1}y^{n-1-j} - \left[\sum_{j=1}^{n-1}x^jy^{n-j}+y^n\right]\\ &= x^n+\sum_{j=0}^{n-2}x^{j+1}y^{n-1-j} - \left[\sum_{k=0}^{n-2}x^{k+1}y^{n-(k+1)}+y^n\right]\\ &\qquad\text{(letting } k=j-1)\\ &= x^n-y^n.\end{aligned}$$

A formal proof requires such a scheme, in which the expression $(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})$ is replaced by the inductively defined symbol $\sum_{j=0}^{n-1}x^jy^{n-1-j}$. Along the way we have used several other manipulations which can, if necessary, be justified by inductive arguments.

**3.** **(iv)** $(a/b)(c/d) = (ab^{-1})(cd^{-1}) = (ac)(b^{-1}d^{-1}) = (ac)(bd)^{-1}$ (by (iii)) $= (ac)/(bd)$.

**(vi)** If $ab^{-1} = cd^{-1}$, then $(ab^{-1})bd = (cd^{-1})bd$, or $ad = bc$. Conversely, if $ad = bc$, then $(ad)d^{-1}b^{-1} = (bc)d^{-1}b^{-1}$, or $ab^{-1} = cd^{-1}$. If $ab^{-1} = ba^{-1}$, then $a^2 = b^2$, so by Problem 1(iii), $a = b$ or $a = -b$. Conversely, if $a = b$, then $a/b = b/a = 1$ and if $a = -b$, then $a/b = b/a = -1$.

**4.** **(ii)** All $x$.

**(iv)** $x > 3$ or $x < 1$.

**(vi)** $x > \left[-1 + \sqrt{5}\right]/2$ or $x < \left[-1 - \sqrt{5}\right]/2$.

**(viii)** All $x$, since $x^2 + x + 1 = [x + (1/2)]^2 + 3/4$.

**(x)** $x > \sqrt{2}$ or $x < \sqrt[3]{2}$.

**(xii)** $x < 1$.

**(xiv)** $x > 1$ or $x < -1$.

**5.** **(ii)** $b - a$ is in $P$, so $-a - (-b)$ is in $P$.

**(iv)** $b - a$ is in $P$ and $c$ is in $P$, so $c(b - a) = bc - ac$ is in $P$.

**(vi)** If $a > 1$, then $a > 0$, so $a^2 > a \cdot 1$, by part (iv).

**(viii)** If $a = 0$ or $c = 0$, then $ac = 0$, but $bd > 0$, so $ac < bd$. Otherwise we have $ac < bc < bd$ by applying part (iv) twice.

**(x)** If $a < b$ were false, then either $a = b$ or $a > b$. But if $a = b$, then $a^2 = b^2$, and if $a > b > 0$, then $a^2 > b^2$, by part (ix).

**6.** **(a)** From $0 < x < y$ and Problem 5(viii) we have $x^2 < y^2$ [as in Problem 5(ix)]. Then from $0 \leq x < y$ and $x^2 < y^2$ we have $x^3 < y^3$. We can continue in this way to prove that $x^n < y^n$ for $n = 2, 3, \dots$ (a rigorous proof uses induction, covered in the next chapter).

**(b)** If $0 \leq x < y$, then $x^n < y^n$ by part (a). If $x < y \leq 0$, then $0 \leq -y < -x$, so $(-y)^n < (-x)^n$ by part (a); this means that $-y^n < -x^n$ (since $n$ is odd) and hence $x^n < y^n$. Finally, if $x < 0 \leq y$, then $x^n < 0 \leq y^n$ (since $n$ is odd). Thus, in all cases, if $x < y$, then $x^n < y^n$.

**(c)** This follows immediately from part (b), since $x < y$ would imply that $x^n < y^n$, while $y < x$ would imply that $y^n < x^n$.

**(d)** Similarly, if $n$ is even, then using part (a) instead of part (b) we see that if $x, y > 0$ and $x^n = y^n$, then $x = y$. Moreover, if $x, y < 0$ and $x^n = y^n$, then $-x, -y \geq 0$ and $(-x)^n = (-y)^n$, so again $x = y$. The only other possibility is that one of $x$ and $y$ is positive, the other negative. In this case $x$ and $-y$ are both positive or both negative. Moreover $x^n = (-y)^n$, since $n$ is even, so it follows from the previous cases that $x = -y$.

**7.** If $a < b$, then
$$a = \frac{a+a}{2} < \frac{a+b}{2} < \frac{b+b}{2} = b.$$
If $0 < a < b$, then $a^2 < ab$ by Problem 5(iv), so $a < \sqrt{ab}$ by Problem 5(x). Moreover, $(a-b)^2 > 0$, so
$$\begin{aligned} a^2 + b^2 &> 2ab, \\ a^2 + 2ab + b^2 &> 4ab, \\ (a+b)^2 &> 4ab, \end{aligned}$$
so $a + b > 2\sqrt{ab}$. Moreover, for all $a, b$ we have $(a - b)^2 \geq 0$, and thus $(a+b)^2 \geq 4ab$, which implies that $a + b \geq 2\sqrt{ab}$ for $a, b \geq 0$.

**8.** Two applications of P$'$12 show that if $a < b$ and $c < d$, then $a+c < b+c < b+d$, so $a + c < b + d$ by P$'$11. In particular, if $0 < b$ and $0 < d$, then $0 < b + d$, which proves P11. It follows, in addition, that if $a < 0$, then $-a > 0$; for if $-a < 0$ were true, then $0 = a + (-a) < 0$, contradicting P$'$10. Consequently, any number $a$ satisfies precisely one of the conditions $a = 0$, $a > 0$, $a < 0$, the last being equivalent to $-a > 0$. This proves P10. Finally, P$'$13 shows that if $0 < a$ and $0 < c$, then $0 < ac$, which proves P12.

**9.** **(ii)** $|a| + |b| - |a + b|$.

**(iv)** $x^2 - 2xy + y^2$.

**10.** (ii)
$$\begin{cases} x - 1 & \text{if } x \geq 1; \\ 1 - x & \text{if } 0 \leq x \leq 1; \\ 1 + x & \text{if } -1 \leq x \leq 0; \\ -1 - x & \text{if } x \leq -1. \end{cases}$$

**(iv)**
$$\begin{aligned} &a \text{ if } a \geq 0; \\ &3a \text{ if } a \leq 0. \end{aligned}$$

**11.** **(ii)** $-5 < x < 11$.

**(iv)** $x < 1$ or $x > 2$ (the distance from $x$ to 1 plus the distance from $x$ to 2 equals 1 precisely when $1 \leq x \leq 2$).

**(vi)** No $x$.

**(viii)** If $x > 1$ or $x < -2$, then the condition becomes $(x - 1)(x + 2) = 3$, or $x^2 + x - 5 = 0$, for which the solutions are $\left(-1 + \sqrt{21}\right)/2$ and $\left(-1 - \sqrt{21}\right)/2$.

Since the first is $> 1$ and the second is $< -2$, both are solutions to the equation $|x-1| \cdot |x+2| = 3$. For $-2 < x < 1$ the condition becomes $(1-x)(x+2) = 3$ or $x^2 + x + 1 = 0$, which has no solutions.

**12.** **(ii)** $|1/x| \cdot |x| = |(1/x) \cdot x|$ (by (i)) $= |1| = 1$, so $|1/x| = 1/|x|$.

**(iv)** $|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|$.

**(vi)** Interchanging $x$ and $y$ in part (v) gives $|y| - |x| \leq |x - y|$. Combining this with part (v) yields $|(|x| - |y|)| \leq |x - y|$.

**13.** If $x \leq y$, then $|y - x| = y - x$, so $x + y + |y - x| = x + y + y - x = 2y$, which is $2\max(x, y)$. Interchanging $x$ and $y$ proves the formula when $x \geq y$, and the same type of argument works for $\min(x, y)$. Also

$$\begin{aligned}\max(x, y, z) &= \max(x, \max(y, z)) \\ &= \frac{x + \dfrac{y + z + |y - z|}{2} + \left|\dfrac{y + z + |y - z|}{2} - x\right|}{2} \\ &= \frac{|y - z| + y + z + 2x + |y + z + |y - z| - 2x|}{4}.\end{aligned}$$

**14.** **(a)** If $a \geq 0$, then $|a| = a = -(-a) = |-a|$, since $-a \leq 0$. The equality is proved for $a \leq 0$ by replacing $a$ by $-a$.

**(b)** If $|a| \leq b$, then clearly $b \geq 0$. Now $|a| \leq b$ means that $a \leq b$ if $a \geq 0$, and surely $a \leq b$ if $a \leq 0$. Similarly, $|a| \leq b$ means $-a \leq b$, and hence $-b \leq a$, if $a \leq 0$, and surely $-b \leq a$ if $a \geq 0$. So $-b \leq a \leq b$.

Conversely, if $-b \leq a \leq b$, then $|a| = a \leq b$ if $a \geq 0$, while $|a| = -a \leq b$ if $a \leq 0$.

**(c)** From $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$ it follows that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|,$$

so $|a + b| \leq |a| + |b|$.

**15.** If $x \neq y$, then

$$x^2 + xy + y^2 = \frac{x^3 - y^3}{x - y}.$$

Problem 6(b) shows that the quotient on the right is always positive (since $x^3 - y^3 > 0$ if $x - y > 0$ and $x^3 - y^3 < 0$ if $x - y < 0$). Moreover, if $x = y \neq 0$, then $x^2 + xy + y^2 = 3x^2 > 0$. The other inequality is proved similarly, using the factorization for $x^5 - y^5$.

16. (a) If

$$x^2 + y^2 = (x + y)^2 = x^2 + 2xy + y^2,$$

then $xy = 0$, so $x = 0$ or $y = 0$. If

$$x^3 + y^3 = (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$

then $3xy(x + y) = 0$, so $x = 0$ or $y = 0$ or $x = -y$.

(b) The first equation implies that

$$4x^2 + 8xy + 4y^2 \geq 0.$$

Suppose that we also had

$$4x^2 + 6xy + 4y^2 \leq 0.$$

Subtracting the second from the first would give $2xy \geq 0$. If neither $x$ nor $y$ is 0, this means that we must have $2xy > 0$; but this implies that $4x^2 + 6xy + y^2 > 0$, a contradiction.

Moreover, it is clear that if one of $x$ and $y$ is 0, but not the other, then we also have $4x^2 + 6xy + 4y^2 > 0$.

(c) If

$$x^4 + y^4 = (x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$$

then

$$0 = 4x^3y + 6x^2y^2 + 4xy^3 = xy(4x^2 + 6xy + 4y^2),$$

so $x = 0$ or $y = 0$, or $4x^2 + 6xy + 4y^2 = 0$. But by part (b), the last equation implies that $x$ and $y$ are both 0. Thus we must always have $x = 0$ or $y = 0$.

(d) If

$$x^5 + y^5 = (x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5,$$

then

$$\begin{aligned} 0 &= 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 \\ &= 5xy(x^3 + 2x^2y + 2xy^2 + y^3), \end{aligned}$$

so $xy = 0$ or

$$x^3 + 2x^2y + 2xy^2 + y^3 = 0.$$

Subtracting this equation from

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

we obtain

$$(x + y)^3 = x^2y + xy^2 = xy(x + y).$$

So either $x+y = 0$ or $(x+y)^2 = xy$; the latter condition implies that $x^2+xy+y^2 = 0$, so $x = 0$ or $y = 0$ by Problem 15. Thus $x = 0$ or $y = 0$ or $x = -y$.

**17.** **(a)** Since

$$\begin{aligned} 2x^2 - 3x + 4 &= 2\left(x - \frac{3}{4}\right)^2 + 4 - \frac{9}{8} \\ &= 2\left(x - \frac{3}{4}\right)^2 + \frac{23}{8}, \end{aligned}$$

the smallest possible value is 23/8, when $(x - 3/4)^2 = 0$, or $x = 3/4$.

**(b)** We have

$$x^2 - 3x + 2y^2 + 4y + 2 = \left(x - \frac{3}{2}\right)^2 + 2(y+1)^2 - \frac{9}{4},$$

so the smallest possible value is $-9/4$, when $x = 3/2$ and $y = -1$.

**(c)** For each $y$ we have

$$\begin{aligned} x^2 + 4xy + 5y^2 - 4x - 6y + 7 &= x^2 + 4(y-1)x + 5y^2 - 6y + 7 \\ &= [x + 2(y-1)]^2 + 5y^2 - 6y + 7 - 4(y-1)^2 \\ &= [x + 2(y-1)]^2 + (y+1)^2 + 2, \end{aligned}$$

so the smallest possible value is 2, when $y = -1$ and $x = -2(y-1) = 4$.

**18.** **(a)** is a straightforward check.

**(b)** We have

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right) \geq c - \frac{b^2}{4};$$

but $c - b^2/4 > 0$, so $x^2 + bx + c > 0$ for all $x$.

**(c)** Apply part (b) with $y$ for $b$ and $y^2$ for $c$: we have $b^2 - 4c = y^2 - 4y^2 < 0$ for $y \neq 0$, so $x^2 + xy + y^2 > 0$ for all $x$, if $y \neq 0$ (and surely $x^2 + xy + y^2 > 0$ for all $x \neq 0$ if $y = 0$).

**(d)** $\alpha$ must satisfy $(\alpha y)^2 - 4y^2 < 0$, or $\alpha^2 < 4$, or $|\alpha| < 2$.

**(e)** Since

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right) \geq c - \frac{b^2}{4},$$

and since $x^2 + bx + c$ has the value $c - b^2/4$ when $x = -b/2$, the minimum value is $c - b^2/4$. Since

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right),$$

the minimum value is

$$a\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)=c-\frac{b^2}{4a}.$$

**19.** **(a)** The proofs when $x_1=\lambda y_1$ and $x_2=\lambda y_2$, or $y_1=y_2=0$, are straightforward. If there is no such $\lambda$, then the equation

$$\lambda^2({y_1}^2+{y_2}^2)-2\lambda(x_1y_1+x_2y_2)+({x_1}^2+{y_1}^2)=0$$

has *no* solution $\lambda$, so by Problem 18(a) we must have

$$\left[\frac{2(x_1y_1+x_2y_2)}{({y_1}^2+{y_2}^2)}\right]^2-\frac{4({x_1}^2+{y_1}^2)}{({y_1}^2+{y_2}^2)}<0,$$

which yields the Schwarz inequality.

**(b)** We have $2xy\le x^2+y^2$, since $0\le(x-y)^2=x^2-2xy+y^2$. Thus

$$(1)\qquad \frac{2x_1y_1}{\sqrt{{x_1}^2+{x_2}^2}\sqrt{{y_1}^2+{y_2}^2}}\le\frac{{x_1}^2}{({x_1}^2+{x_2}^2)}+\frac{{y_1}^2}{({y_1}^2+{y_2}^2)},$$

$$(2)\qquad \frac{2x_1y_1}{\sqrt{{x_1}^2+{x_2}^2}\sqrt{{y_1}^2+{y_2}^2}}\le\frac{{x_2}^2}{({x_1}^2+{x_2}^2)}+\frac{{y_2}^2}{({y_1}^2+{y_2}^2)};$$

addition yields

$$\frac{2(x_1y_1+x_2y_2)}{\sqrt{{x_1}^2+{x_2}^2}\sqrt{{y_1}^2+{y_2}^2}}\le 2.$$

**(c)** The equality is a straightforward computation. Since $(x_1y_2-x_2y_1)^2\ge 0$, the Schwarz inequality follows immediately.

**(d)** The proof in part (a) already yields the desired result.

In part (b), equality holds only if it holds in (1) and (2). Since $2xy=x^2+y^2$ only when $0=(x-y)^2$, i.e., $x=y$, this means that

$$\frac{x_i}{\sqrt{{x_1}^2+{x_2}^2}}=\frac{y_i}{\sqrt{{y_1}^2+{y_2}^2}}\qquad\text{for } i=1,2,$$

so we can choose $\lambda=\sqrt{{x_1}^2+{x_2}^2}\big/\sqrt{{y_1}^2+{y_2}^2}$.

In part (c), equality holds only when $x_1y_2-x_2y_1=0$. One possibility is $y_1=y_2=0$. If $y_1\ne 0$, then $x_1=(x_1/y_1)y_1$ and also $x_1=(x_1/y_1)y_2$; similarly, if $y_2=0$, then $\lambda=x_2/y_2$.

**20, 21, 22.** See Chapter 5.

**23.** According to Problem 21, we have $|x/y-x_0/y_0|<\varepsilon$ if

$$|x-x_0|<\min\left(\frac{\varepsilon}{2(1/|y_0|+1)},1\right)$$

and

$$\left|\frac{1}{y}-\frac{1}{y_0}\right| < \frac{\varepsilon}{2(|x_0|+1)},$$

and the latter is true, according to Problem 22, if

$$|y-y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon|y_0|^2}{4(|x_0|+1)}\right).$$

**24.** **(a)** For $k=1$ the equation reads $a_1+a_2=a_1+a_2$. If the equation holds for $k$, then

$$\begin{aligned}(a_1+\cdots+a_{k+1})+a_{k+2} &= [(a_1+\cdots+a_k)+a_{k+1}]+a_{k+2}\\ &= (a_1+\cdots+a_k)+(a_{k+1}+a_{k+2}) \qquad \text{by P1}\\ &= a_1+\cdots+a_k+(a_{k+1}+a_{k+2})\\ &\qquad\qquad \text{since the equation holds for } k\\ &= a_1+\cdots+a_{k+2}\\ &\qquad\qquad \text{by the definition of } a_1+\cdots+a_{k+2}.\end{aligned}$$

**(b)** For $k=1$ the equation reduces to the definition of $a_1+\cdots+a_k$. If the equation is true for some $k<n$, then

$$\begin{aligned}(a_1+\cdots+a_{k+1})&+(a_{k+2}+\cdots+a_n)\\ &= ([a_1+\cdots+a_k]+a_{k+1})+(a_{k+2}+\cdots+a_n)\\ &\qquad \text{by part (a)}\\ &= (a_1+\cdots+a_k)+(a_{k+1}+(a_{k+2}+\cdots+a_n))\\ &\qquad \text{by P1}\\ &= (a_1+\cdots+a_k)+(a_{k+1}+\cdots+a_n)\\ &\qquad \text{by the definition of } a_{k+1}+\cdots+a_n\\ &= a_1+\cdots+a_n \qquad \text{by assumption.}\end{aligned}$$

**(c)** The proof is by "complete induction" on $k$ (see Chapter 2). The assertion is clear for $k=1$. Assume that it is true for all $l<k$. Then

$$\begin{aligned}s(a_1,\ldots,a_k) &= s'(a_1,\ldots,a_l)+s''(a_{l+1},\ldots,a_k)\\ &= (a_1+\cdots+a_l)+(a_{l+1}+\cdots+a_k) \qquad \text{by assumption}\\ &= a_1+\cdots+a_k \qquad \text{by part (b).}\end{aligned}$$

**25.** P2, P3, P4, P6, P7, P8 are obvious from a glance at the tables. There are eight cases for P1, and even this number can be reduced: because P2 is true, it is clear that $a+(b+c)=(a+b)+c$ if $a$, $b$, or $c$ is 0, so only the case $a=b=c=1$ must be checked. Similarly for P5. Finally, P9 is true for $a=0$, since $0\cdot b=0$ for all $b$, and for $a=1$, since $1\cdot b=b$ for all $b$.

**1.** (ii) Since $1^3 = 1^2$, the formula is true for $n = 1$. Suppose that the formula is true for $k$. Then

$$\begin{aligned}(1 + \cdots + k + [k+1])^2 &= (1 + \cdots + k)^2 + 2(1 + \cdots + k)(k+1) + (k+1)^2 \\ &= 1^3 + \cdots + k^3 + 2\frac{k(k+1)}{2}(k+1) + (k+1)^2 \\ &= 1^3 + \cdots + k^3 + (k^3 + 2k^2 + k) + (k^2 + 2k + 1) \\ &= 1^3 + \cdots + k^3 + (k+1)^3,\end{aligned}$$

so the formula is true for $k + 1$.

**2.** (ii)

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)^2 &= 1^2 + 3^2 + \cdots + (2n-1)^2 \\ &= [1^2 + 2^2 + \cdots + (2n)^2] - [2^2 + 4^2 + 6^2 + \cdots + (2n)^2] \\ &= [1^2 + 2^2 + \cdots + (2n)^2] - 4[1^2 + 2^2 + 3^2 + \cdots + (n)^2] \\ &= \frac{2n(2n+1)(4n+1)}{6} - \frac{4n(n+1)(2n+1)}{6} \\ &= \frac{2n(2n+1)[4n+1-2(n+1)]}{6} \\ &= \frac{n(2n+1)(2n-1)}{3}.\end{aligned}$$

**3.** (a)

$$\begin{aligned}\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\ &= \frac{kn!}{k!(n+1-k)!} + \frac{(n+1-k)n!}{k!(n+1-k)!} \\ &= \frac{(n+1)n!}{k!(n+1-k)!} = \binom{n+1}{k}.\end{aligned}$$

(b) Clearly $\binom{1}{1}$ is a natural number. Suppose that $\binom{n}{p}$ is a natural number for all $p < n$. Since

$$\binom{n+1}{p} = \binom{n}{p-1} + \binom{n}{p} \qquad \text{for } p \leq n.$$

it follows that $\binom{n+1}{p}$ is a natural number for all $p \leq n$, while $\binom{n+1}{n+1}$ is also a natural number. So $\binom{n+1}{p}$ is a natural number for all $p \leq n+1$.

**(c)** There are $n(n-1)\cdots(n-k+1)$ $k$-tuples of distinct integers each chosen from $1,\ldots,n$, since the first can be picked in $n$ ways, the next in $n-1$ ways, etc. Now each set of exactly $k$ integers can be arranged in $k!$ $k$-tuples, so there are $n(n-1)\cdots(n-k+1)/k! = \binom{n}{k}$ such sets.

**(d)** The binomial theorem is clear for $n=1$. Suppose that

$$(a+b)^n = \sum_{j=0}^{n}\binom{n}{j}a^{n-j}b^j.$$

Then

$$\begin{aligned}(a+b)^{n+1} &= (a+b)(a+b)^n = (a+b)\sum_{j=0}^{n}\binom{n}{j}a^{n-j}b^j\\ &= \sum_{j=0}^{n}\binom{n}{j}a^{n+1-j}b^j + \sum_{j=0}^{n}\binom{n}{j}a^{n-j}b^{j+1}\\ &= \sum_{j=0}^{n}\binom{n}{j}a^{n+1-j}b^j + \sum_{j=1}^{n+1}\binom{n}{j-1}a^{n+1-j}b^j\\ &\qquad\text{(we have replaced } j \text{ by } j-1 \text{ in the second sum)}\\ &= \sum_{j=0}^{n+1}\binom{n+1}{j}a^{n+1-j}b^j \qquad \text{by part (a),}\end{aligned}$$

so the binomial theorem is true for $n+1$.

**(e)** **(i)**

$$2^n = (1+1)^n = \sum_{j=0}^{n}\binom{n}{j}.$$

**(ii)**

$$0 = (1+-1)^n = \sum_{j=0}^{n}(-1)^j\binom{n}{j}.$$

**(iii)** Subtracting (ii) from (i) we obtain

$$2\sum_{l \text{ odd}}\binom{n}{l} = 2^n.$$

**(iv)** Add (i) and (ii).

**4.** **(a)** Since

$$(1+x)^n(1+x)^m = (1+x)^{n+m}$$

we have

$$\sum_{k=0}^{n}\binom{n}{k}x^k \cdot \sum_{j=0}^{m}\binom{m}{j}x^j = \sum_{l=0}^{n+m}\binom{n+m}{l}x^l.$$

But the coefficient of $x^l$ on the left is clearly

$$\sum_{k=0}^{l} \binom{n}{k}\binom{m}{l-k},$$

one term of the sum occurring for each pair $k$, $j = l - k$.

(b) Let $m, l = n$ in part (a) [note that $\binom{n}{k} = \binom{n}{n-k}$].

6. (ii) From

$$(k+1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1 \qquad k = 1, \ldots, n$$

we obtain

$$(n+1)^5 - 1 = 5\left(\sum_{k=1}^{n} k^4\right) + 10\left(\sum_{k=1}^{n} k^3\right) + 10\left(\sum_{k=1}^{n} k^2\right) + 5\left(\sum_{k=1}^{n} k\right) + n,$$

so

$$\sum_{k=1}^{n} k^4 = \frac{(n+1)^5 - 1 - 10\left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right) - 10\frac{n(n+1)(n+2)}{6} - 5\frac{n(n+1)}{2} - n}{5}$$

$$= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

(iv) From

$$\frac{1}{k^2} - \frac{1}{(k+1)^2} = \frac{2k+1}{k^2(k+1)^2} \qquad k = 1, \ldots, n$$

we obtain

$$1 - \frac{1}{(n+1)^2} = \sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2}.$$

7. The proof is by complete induction on $p$. The statement is true for $p = 1$, since

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{n^2}{2} + n.$$

Suppose that the statement is true for all natural numbers $\leq p$. The binomial theorem yields the equations

$$(k+1)^{p+1} - k^{p+1} = (p+1)k^p + \text{terms involving lower powers of } k.$$

Adding for $k = 1, \ldots, n$, we obtain

$$\frac{(n+1)^{p+1}}{p+1} = \sum_{k=1}^{n} k^p + \text{terms involving } \sum_{k=1}^{n} k^r \text{ for } r < p.$$

By assumption, we can write each $\sum_{k=1}^{n} k^r$ as an expression involving powers $n^s$ with $s \le p$. It follows that

$$\sum_{k=1}^{n} k^p = \frac{(n+1)^{p+1}}{p+1} + \text{terms involving powers of } n \text{ less thatn } p+1.$$

**10.** Suppose $A$ contains 1, and that $A$ contains $n+1$ if it contains $n$. If $A$ does not contain all natural numbers, then the set $B$ of natural numbers *not* in $A$ is not $\emptyset$. So $B$ has a smallest member $n_0$. Now $n_0 \neq 1$, since $A$ contains 1, so we can write $n_0 = (n_0 - 1) + 1$, where $n_0 - 1$ is a natural number. Now $n_0 - 1$ is *not* in $B$, so $n_0 - 1$ *is* in $A$. By hypothesis, $n_0$ must be in $A$, so $n_0$ is not in $B$, a contradiction. (By the way, the assertion that a natural number $n \neq 1$ can be written $n = m + 1$ for some other natural number $m$, can itself be proved by induction.)

**11.** Clearly 1 is in $B$. If $k$ is in $B$, then $1, \dots, k$ are all in $A$, so $k+1$ is in $A$, so $1, \dots, k+1$ are in $A$, so $k+1$ is in $B$. By (ordinary) induction, $B = \mathbf{N}$, so also $A = \mathbf{N}$.

**14.** (a) If $\sqrt{2} + \sqrt{6}$ were rational, then $\left(\sqrt{2} + \sqrt{6}\right)^2$ would certainly be rational. So $8 + 2\sqrt{12} = 8 + 4\sqrt{3}$ would be rational, so $\sqrt{3}$ would be rational, which is false.

(b) Similarly, if $\sqrt{2} + \sqrt{3}$ were rational, then its square $5 + 2\sqrt{6}$ would be rational, so $\sqrt{6}$ would be rational, which is false.

**15.** (a) The assertion is true for $m = 1$. If it is true for $m$, then

$$(p + \sqrt{q}\,)^{m+1} = (p + \sqrt{q}\,)(a + b\sqrt{q}\,) = (ap + bq) + (a + pb)\sqrt{q},$$

and $ap + bq$ and $a + bp$ are rational.

(b) The assertion is true for $m = 1$. If it is true for $m$, then

$$(p - \sqrt{q}\,)^{m+1} = (p - \sqrt{q}\,)(a - b\sqrt{q}\,) = (ap + bq) - (a + pb)\sqrt{q},$$

whereas $(p + \sqrt{q}\,)^{m+1} = (ap + bq) + (a + pb)\sqrt{q}$ by part (a).

**16.** (a) The inequality $(m + 2n)^2/(m + n)^2 > 2$ is equivalent to

$$m^2 + 4mn + 4n^2 > 2m^2 + 4mn + 2n^2,$$

or simply $2n^2 > m^2$.

The second inequality is equivalent to

$$n^2[(m + 2n)^2 - 2(m + n)^2] < (2n^2 - m^2)(m + n)^2,$$

or

$$n^2(2n^2 - m^2) < (2n^2 - m^2)(n^2 + [2mn + m^2]).$$

or

$$0 < (2n^2 - m^2)(2mn + m^2).$$

(b) Reverse all inequality signs in the solution for part (a).

(c) Let $m_1 = m + 2n$ and $n_1 = m + n$, and then choose

$$\begin{aligned} m' &= m_1 + 2n_1 = 3m + 4n, \\ n' &= m_1 + \ \ n_1 = 2m + 3n. \end{aligned}$$

17. (a) Suppose that every number $< n$ can be written as a product of primes. If $n > 1$ is not a prime, then $n = ab$ for $a, b < n$. By assumption, $a$ and $b$ are each products of primes, so $n = ab$ is also.

(b) If $\sqrt{n} = a/b$, then $nb^2 = a^2$, so the factorization into primes of $nb^2$ and of $a^2$ must be the same. Now every prime appears an even number of times in the factorization of $a^2$, and of $b^2$, so the same must be true of the factorization of $n$. This implies that $n$ is a square.

(c) Repeat the same argument, using the fact that every prime occurs a multiple of $k$ times in $a^k$ and $b^k$.

(d) If $p_1, \ldots, p_n$ were the only primes, then $(p_1 \cdot p_2 \cdots p_n) + 1$ could not be a prime, since it is larger than all of them (and is not 1), so it must be divisible by a prime. But $p_1, \ldots, p_n$ clearly do not divide it, a contradiction. (Although this is a proof by contradiction, it can be used to obtain some positive information: If $p_1, \ldots, p_n$ are the first $n$ primes, then the $(n+1)^{st}$ prime is $\le (p_1 \cdot p_2 \cdots p_n) + 1$. It is not necessarily true, however, that the number $(p_1 \cdot p_2 \cdots p_n) + 1$ is a prime; for example, $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13) + 1 = 30{,}031 = 59 \cdot 509$.)

18. (a) Suppose $x = p/q$ where $p$ and $q$ are natural numbers with no common factor. Then

$$\frac{p^n}{q^n} + a_{n-1}\frac{p^{n-1}}{q^{n-1}} + \cdots + a_0 = 0,$$

so

$$(*) \qquad p^n + a_{n-1}p^{n-1}q + \cdots + a_0q^n = 0.$$

Now if $q \neq \pm 1$, then $q$ has some prime number as a factor. This prime factor divides every term of $(*)$ other than $p^n$, so it must divide $p^n$ also. Therefore it divides $p$, a contradiction. So $q = \pm 1$, which means that $x$ is an integer.

(b) If

$$x = \sqrt{6} - \sqrt{2} - \sqrt{3},$$

then

$$\begin{aligned} x^2 &= 6 + (\sqrt{2}+\sqrt{3})^2 - 2\sqrt{6}(\sqrt{2}+\sqrt{3}) \\ &= 11 + 2\sqrt{6}\left[1 - (\sqrt{2}+\sqrt{3})\right], \end{aligned}$$

so

$$\begin{aligned} (x^2-11)^2 &= 24\left[1 - (\sqrt{2}+\sqrt{3})\right]^2 \\ &= 24\left[1 + (\sqrt{2}+\sqrt{3})^2 - 2(\sqrt{2}+\sqrt{3})\right] \\ &= 24\left[6 + 2(\sqrt{6}-\sqrt{2}-\sqrt{3})\right] \\ &= 24[6+2x]. \end{aligned}$$

It follows from part (a) that either $x$ is irrational or else $x$ is an integer. But it is easy to check that

$$0 < \sqrt{2}+\sqrt{3}-\sqrt{6} < 1$$

(the inequalities $\sqrt{6} < \sqrt{2}+\sqrt{3}$ and $\sqrt{2}+\sqrt{3} < 1+\sqrt{6}$ are easily checked by squaring them), so $x$ is not an integer.

(c) Writing the various powers of $x = 2^{2/6}+2^{3/6}$ in terms of the powers of $\eta = 2^{1/6}$, we obtain the following table for the coefficients.

| | $\eta^0$ | $\eta^1$ | $\eta^2$ | $\eta^3$ | $\eta^4$ | $\eta^5$ |
|---|---|---|---|---|---|---|
| $x^0$ | 1 | | | | | |
| $x^1$ | | | 1 | 1 | | |
| $x^2$ | 2 | | | | 1 | 2 |
| $x^3$ | 2 | 6 | 6 | 2 | | |
| $x^4$ | | | 2 | 8 | 12 | 8 |
| $x^5$ | 40 | 40 | 20 | 4 | 2 | 10 |
| $x^6$ | 12 | 24 | 60 | 80 | 60 | 24 |

We can then find numbers $a_0, \dots, a_5$ such that

$$x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

by solving the equations $a_0 + 2a_2 + 2a_3 + 40a_5 + 12 = 0$, etc. It turns out that

$$x^6 - 6x^4 - 4x^3 + 12x^2 - 24x - 4 = 0.$$

Part (a) implies that either $x$ is irrational or else $x$ is an integer, and it is easy to see that $x$ is not an integer, because $1.4 < \sqrt{2} < 1.5$ and $1.2 < \sqrt[3]{2} < 1.3$, so $2.6 < \sqrt{2}+\sqrt[3]{2} < 2.8$.

This is one of those problems where a little learning, though perhaps a dangerous thing, could save a lot of work: The proper equation for $x$ can also be found by noting that $\sqrt{2} + \sqrt[3]{2}$ clearly satisfies the equation

$$\left[\left(x - \sqrt{2}\right)^3 - 2\right] \cdot \left[\left(x + \sqrt{2}\right)^3 - 2\right] = 0;$$

when the left side is multiplied out we obtain

$$\begin{aligned}
&(x^2 - 2)^3 + 4 - 2 \cdot \left[\left(x - \sqrt{2}\right)^3 + \left(x + \sqrt{2}\right)^3\right] \\
&\quad = (x^2 - 2)^3 + 4 - 2 \cdot [2x^3 + 12x] \qquad \text{(the odd powers of } x \text{ cancel out)} \\
&\quad = x^6 - 6x^4 - 4x^3 + 12x^2 - 24x - 4.
\end{aligned}$$

Of course, this method depends on the observation that the equation for $x = \sqrt{2} + \sqrt[3]{2}$ should also have $-\sqrt{2} + \sqrt[3]{2}$ as a root (a hint as to why this should be true will be found in Problem 25-8).

**20.** Since

$$\frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^1 - \left(\dfrac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1,$$

$$\frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^2 - \left(\dfrac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1,$$

the assertion is true for $n = 1$ and $n = 2$. Now suppose that the assertion is true for all $k < n$, where $n \geq 3$. Then it is true, in particular, for $n - 1$ and $n - 2$, so

$$\begin{aligned}
a_n &= a_{n-1} + a_{n-2} \\
&= \frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^{n-2} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n-2} + \left(\dfrac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}} \\
&= \frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^{n-2}\left(1 + \dfrac{1+\sqrt{5}}{2}\right) - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n-2}\left(1 + \dfrac{1-\sqrt{5}}{2}\right)}{\sqrt{5}} \\
&= \frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^{n-2}\left(\dfrac{1+\sqrt{5}}{2}\right)^2 - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n-2}\left(\dfrac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} \\
&= \frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.
\end{aligned}$$

**21.** **(a)** As before, the proof is trivial if all $y_i = 0$ or if there is some number $\lambda$ with $x_i = \lambda y_i$ for all $i$. Otherwise,

$$\begin{aligned} 0 &< \sum_{i=1}^{n}(\lambda y_i - x_i)^2 \\ &= \lambda^2\left(\sum_{i=1}^{n} y_i{}^2\right) - 2\lambda\left(\sum_{i=1}^{n} x_i y_i\right) + \sum_{i=1}^{n} x_i{}^2, \end{aligned}$$

so Problem 1-18 again gives the result.

**(b)** Using $2xy \le x^2 + y^2$ with

$$x = \frac{x_i}{\sqrt{\sum_{i=1}^{n} x_i{}^2}}, \qquad y = \frac{y_i}{\sqrt{\sum_{i=1}^{n} y_i{}^2}}$$

we obtain

$$(1) \qquad \frac{2x_i y_i}{\sqrt{\sum_{i=1}^{n} x_i{}^2}\sqrt{\sum_{i=1}^{n} y_i{}^2}} \le \frac{x_i{}^2}{\sum_{i=1}^{n} x_i{}^2} + \frac{y_i{}^2}{\sum_{i=1}^{n} y_i{}^2}.$$

Adding we obtain

$$\frac{\sum_{i=1}^{n} 2x_i y_i}{\sqrt{\sum_{i=1}^{n} x_i{}^2}\sqrt{\sum_{i=1}^{n} y_i{}^2}} \le \frac{\sum_{i=1}^{n} x_i{}^2}{\sum_{i=1}^{n} x_i{}^2} + \frac{\sum_{i=1}^{n} y_i{}^2}{\sum_{i=1}^{n} y_i{}^2} = 2.$$

Again, equality holds only if it holds in (1) for all $i$, which means that

$$\frac{x_i}{\sqrt{\sum_{i=1}^{n} x_i{}^2}} = \frac{y_i}{\sqrt{\sum_{i=1}^{n} y_i{}^2}}$$

for all $i$. If all $y_i$ are not 0, this means that $x_i = \lambda y_i$ for

$$\lambda = \frac{\sqrt{\sum_{i=1}^{n} x_i{}^2}}{\sqrt{\sum_{i=1}^{n} y_i{}^2}}.$$

(c) This is the most interesting proof—it depends on the equality

$$\sum_{i=1}^{n} x_i^2 \cdot \sum_{i=1}^{n} y_i^2 = \left(\sum_{i=1}^{n} x_i y_i\right)^2 + \sum_{i<j} (x_i y_j - x_j y_i)^2.$$

To check this equality, note that

$$\sum_{i=1}^{n} x_i^2 \cdot \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} x_i^2 y_i^2 + \sum_{i \neq j} x_i^2 y_j^2$$

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 = \sum_{i=1}^{n} (x_i y_i)^2 + \sum_{i \neq j} x_i y_i x_j y_j.$$

The difference is

$$\begin{aligned}\sum_{i \neq j} (x_i^2 y_j^2 - x_i y_i x_j y_j) &= 2\sum_{i<j} (x_i^2 y_j^2 + x_j^2 y_i^2 - x_i y_i x_j y_j) \\ &= 2\sum_{i<j} (x_i y_j - x_j y_i)^2.\end{aligned}$$

If equality holds in the Schwarz inequality, then all $x_i y_j = x_j y_i$. If some $y_i \neq 0$, say $y_1 \neq 0$, then $x_i = \dfrac{x_1}{y_1} y_i$ for all $i$, so we can let $\lambda = x_1/y_1$.

**22.** (a) We have to prove that

$$A_n(a_1 + a_2 - A_n) \geq a_1 a_2$$

or

$$\begin{aligned}0 &\geq A_n^2 - (a_1 + a_2)A_n + a_1 a_2 \\ &= (A_n - a_1)(A_n - a_2),\end{aligned}$$

which is indeed true, since $a_1 < A_n < a_2$. In fact, we actually have $\bar{a}_1\bar{a}_2 > a_1 a_2$.

This shows that $G_n \leq \bar{G}_n$, the geometric mean of $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$, while the arithmetic mean $\bar{A}_n$ is the same as $A_n$. So it suffices to prove that $\bar{G}_n < \bar{A}_n = A_n$. In other words, we can assume that one of the numbers (namely $\bar{a}_1$) actually equals the arithmetic mean. But now we can repeat this process and see that it suffices to prove the inequality when *two* of the numbers equal the arithmetic mean. Continuing enough times, it suffices to prove the inequality when all numbers are equal, in which case it is clearly true, and in fact, is an equality. This is clearly the only case where we have equality, since at the very first stage we get $G_n < \bar{G}_n$ if some $a_i \neq A_n$.

(b) We know that $G_n \le A_n$ when $n = 2^1$. Suppose that $G \le A_n$ for $n = 2^k$ and let $m = 2^{k+1} = 2n$. Then

$$
\begin{aligned}
G_m &= \sqrt[m]{a_1 \cdots a_m} = \sqrt{\sqrt[n]{a_1 \cdots a_n}\,\sqrt[n]{a_{n+1} \cdots a_m}} \\
&\le \frac{\sqrt[n]{a_1 \cdots a_n} + \sqrt[n]{a_{n+1} \cdots a_m}}{2} && \text{using } G_2 \le A_2 \\
&\le \frac{\dfrac{a_1 + \cdots + a_n}{n} + \dfrac{a_{n+1} + \cdots + a_m}{n}}{2} && \text{by assumption} \\
&= \frac{a_1 + \cdots + a_m}{2n} \\
&= A_m.
\end{aligned}
$$

(c) Applying (b) to these $2^m$ numbers yields, for $k = 2^m - n$,

$$
\begin{aligned}
(a_1 \cdots a_n)(A_n)^k &\le \left[\frac{a_1 + \cdots + a_n + kA_n}{2^m}\right]^{2^m} \\
&= \left[\frac{nA_n + kA_n}{2^m}\right]^{2^m} = (A_n)^{2^m},
\end{aligned}
$$

so

$$a_1 \cdots a_n \le (A_n)^{2^m - k} = (A_n)^n.$$

**23.** Since $a^{n+1} = a^n \cdot a = a^n \cdot a^1$, the first equation is true for $m = 1$. Suppose that $a^{n+m} = a^n \cdot a^m$. Then

$$
\begin{aligned}
a^{n+(m+1)} &= a^{(n+m)+1} = a^{n+m} \cdot a && \text{by definition} \\
&= (a^n \cdot a^m) \cdot a \\
&= a^n \cdot (a^m \cdot a) \\
&= a^n \cdot a^{m+1} && \text{by definition,}
\end{aligned}
$$

so the first equation is true for $m + 1$.

Since $(a^n)^1 = a^n = a^{n \cdot 1}$, the second equation is true for $m = 1$. Suppose that $(a^n)^m = a^{nm}$. Then

$$
\begin{aligned}
(a^n)^{m+1} &= (a^n)^m \cdot a^n && \text{by definition} \\
&= a^{nm} \cdot a^n \\
&= a^{nm+n} && \text{by (i)} \\
&= a^{n(m+1)}.
\end{aligned}
$$

**24.** Since

$$\begin{aligned} 1 \cdot (b+c) &= b+c && \text{by definition} \\ &= 1 \cdot b + 1 \cdot c && \text{by definition,} \end{aligned}$$

the first result is true for $a = 1$. Suppose that $a \cdot (b+c) = a \cdot b + a \cdot c$ for all $b$ and $c$. Then

$$\begin{aligned} (a+1) \cdot (b+c) &= a \cdot (b+c) + (b+c) && \text{by definition} \\ &= (a \cdot b + a \cdot c) + (b+c) \\ &= (a \cdot b + b) + (a \cdot c + c) && \text{by P1 and P4} \\ &= (a+1) \cdot b + (a+1) \cdot c && \text{by definition.} \end{aligned}$$

The equation $a \cdot 1 = a$ is true for $a = 1$ by definition. Suppose that $a \cdot 1 = a$. Then

$$\begin{aligned} (a+1) \cdot 1 &= a \cdot 1 + 1 \cdot 1 && \text{by definition} \\ &= a + 1. \end{aligned}$$

For $b = 1$, the equation $a \cdot b = b \cdot a$ follows from $a \cdot 1 = a$, which has just been proved, and $1 \cdot a = a$, which is true by definition. Suppose that $a \cdot b = b \cdot a$. Then

$$\begin{aligned} a \cdot (b+1) &= a \cdot b + a \cdot 1 \\ &= a \cdot b + a \\ &= b \cdot a + a \\ &= (b+1) \cdot a && \text{by definition.} \end{aligned}$$

**25.** **(a)** **(i)** is clear.

**(ii)** This is clear, because 1 is positive, and if $k$ is positive, then $k + 1$ is positive.

**(iii)** Clearly 1 is in this set. If condition (2) failed for this set, then there would be some $k$ in the set with $k + 1 = 1/2$. But his is false, since $k = -1/2$ is not positive.

**(iv)** This set contains 4 but not $4 + 1$.

**(v)** Since 1 is in $A$ and $B$, also 1 is in $C$. If $k$ is in $C$, then $k$ is in both $A$ and $B$, so $k + 1$ is in $A$ and $B$, so $k + 1$ is in $C$.

**(b)** **(i)** 1 is a natural number because 1 is in every inductive set, by definition of inductive sets.

**(ii)** If $k$ is a natural number, then $k$ is in every inductive set. So $k + 1$ is in every inductive set. So $k + 1$ is a natural number.

**26.** If there is only $n = 1$ ring, it can clearly be moved onto spindle 3 in $1 = 2^1 - 1$ moves. Assume the result for $k$ rings. Then given $k + 1$ rings,

(a) move the top $k$ rings onto spindle 2 in $2^k - 1$ moves,

(b) move the bottom ring onto spindle 3,

(c) move the top $k$ rings back onto spindle 3 in $2^k - 1$ moves.

This takes $2(2^k - 1) + 1 = 2^{k+1} - 1$ moves. If $2^k - 1$ moves is the minimum possible for $k$ rings, then $2^{k+1} - 1$ is the minimum for $k + 1$ rings, since the bottom ring can't be moved at all until the top $k$ rings are moved somewhere, taking at least $2^k - 1$ moves, the bottom ring has to be moved to spindle 3, taking at least 1 move, and then the other rings have to be placed on top of it, taking at least another $2^k - 1$ moves.

**27.** Everyone resigned on the seventeenth luncheon meeting.

The reasoning is as follows (for the sake of sanity, "he or she" shall be rendered as "he" throughout). First suppose there were only 2 professors, Prof. A and Prof. B, each knowing of the error in the other's work, but unaware of any error in his own. Then neither is surprised by Prof. X's statement, but each *expects the other to be surprised*, and to resign at the first luncheon meeting next year. When this doesn't happen, each (being a mathematics professor capable of logical deduction) realizes that this can only be because he has also made an error. So at the next meeting, both resign.

Next consider the case of 3 professors, Profs. A, B and C. Prof. C knows that Prof. A is aware of an error in Prof. B's work (either because Prof. A found the error and informed him, or because he found the error and informed Prof. A). Similarly, he knows that Prof. B knows that there is an error in Prof. A's work. But Prof. C thinks he has made no errors, so as far as he is concerned, the situation vis-a-vis Profs. A and B is precisely that analyzed in the previous paragraph (Prof. C is assuming, of course, that no one believes an error to exist when one doesn't). So Prof. C expects both Prof. A and Prof. B to resign at the second meeting. Of course, Profs. A and B similarly expect the other two to resign at the second meeting. When no one resigns, everyone realizes that he has made an error, so all resign at the third meeting.

Now you can turn this into a proof by induction (can't you?).

**28.** Again it is a good idea to start with the case when the department consists only of Profs. A and B. Now, of course, both professors know that some one has published an incorrect result, but Prof. A thinks that Prof. B *doesn't* know, and vice-versa. Once Prof. X makes his announcement, Prof. A knows that Prof. B knows. And that's why he expects Prof. B to resign at the next meeting.

In the case of three professors, the situation is more complicated. Each knows that some one has made an error, and moreover each knows that the others know—for example, Prof. C knows that Prof. A knows, since he and Prof. A have discussed the error in Prof. B's work, and he knows similarly that Prof. B knows. But Prof. C *doesn't* think that Prof. A knows that Prof. B knows. So Prof. X's announcement changes things: now Prof. C knows that Prof. A knows that Prof. B knows.

Well, you can see what happens in general. This seems to prove that statements like "A knew that B knew that C knew that . . . " actually make sense.