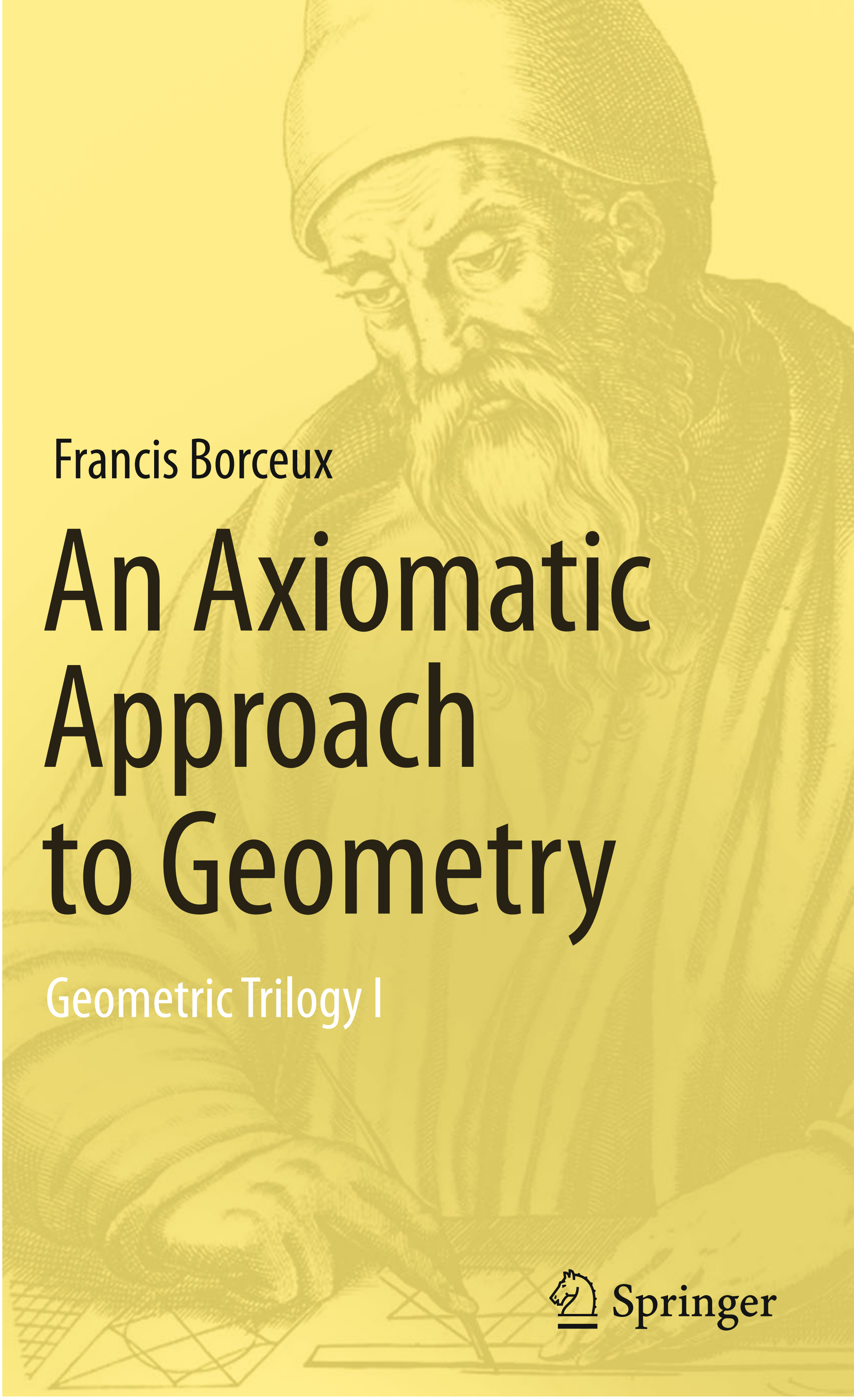

Francis Borceux

An Axiomatic Approach to Geometry

Geometric Trilogy I

Springer

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Francis Borceux
Université catholique de Louvain
Louvain-la-Neuve, Belgium

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To Christiane

Preface

The reader is invited to immerse himself in a "love story" which has been unfolding for 35 centuries: the love story between mathematicians and geometry. In addition to accompanying the reader up to the present state of the art, the purpose of this *Trilogy* is precisely to tell this story. The *Geometric Trilogy* will introduce the reader to the multiple complementary aspects of geometry, first paying tribute to the historical work on which it is based and then switching to a more contemporary treatment, making full use of modern logic, algebra and analysis. In this *Trilogy*, Geometry is definitely viewed as an autonomous discipline, never as a sub-product of algebra or analysis. The three volumes of the *Trilogy* have been written as three independent but complementary books, focusing respectively on the axiomatic, algebraic and differential approaches to geometry. They contain all the useful material for a wide range of possibly very different undergraduate geometry courses, depending on the choices made by the professor. They also provide the necessary geometrical background for researchers in other disciplines who need to master the geometric techniques.

The present book leads the reader on a walk through 35 centuries of geometry: from the papyrus of the Egyptian scribe *Ahmes*, 16 centuries before Christ, to Hilbert's famous axiomatization of geometry, 19 centuries after Christ. We discover step by step how all the ingredients of contemporary geometry have slowly acquired their final form.

It is a matter of fact: for three millennia, geometry has essentially been studied via "synthetic" methods, that is, from a given system of axioms. It was only during the 17th century that algebraic and differential methods were considered seriously, even though they had always been present, in a disguised form, since antiquity.

After rapidly reviewing some results that had been known empirically by the Egyptians and the Babylonians, we show how Greek geometers of antiquity, slowly, sometimes encountering great difficulties, arrived at a coherent and powerful deductive theory allowing the rigorous proof of all of these empirical results, and many others. Famous problems—such as "squaring the circle"—induced the development of sophisticated methods. In particular, during the fourth century BC, *Eudoxus* overcame the puzzling difficulty of "incommensurable quantities" by a method which is essentially that of Dedekind cuts for handling real numbers. Eudoxus also proved the validity of a "limit process" (the *Exhaustion theorem*) which allowed him to answer questions concerning, among other things, the lengths, areas or volumes related to various curves or surfaces.

We first summarize the knowledge of the Greek geometers of the time by presenting the main aspects of *Euclid's Elements*. We then switch to further work by *Archimedes* (the circle, the sphere, the spiral, ...), *Apollonius* (the conics), *Menelaus* and *Ptolemy* (the birth of trigonometry), *Pappus* (ancestor of projective geometry), and so on.

We also review some relevant results of classical *Euclidean geometry* which were only studied several centuries after Euclid, such as additional properties of triangles and conics, Ceva's theorem, the trisectors of a triangle, stereographic projection, and so on. However, the most important new aspect in this spirit is probably the theory of inversions (a special case of a conformal mapping) developed by Poncelet during the 19th century.

We proceed with the study of projective methods in geometry. These appeared in the 17th century and had their origins in the efforts of some painters to understand the rules of perspective. In a good perspective representation, parallel lines seem to meet "at the horizon". From this comes the idea of adding "points at infinity" to the Euclidean plane, points where parallel line eventually meet. For a while, projective methods were considered simply as a convenient way to handle some Euclidean problems. The recognition of projective geometry as a respectable geometric theory in itself—a geometry where two lines in the plane always intersect—only came later. After having discussed the fundamental ideas which led to projective geometry—we focus on the amazing *Hilbert theorems*. These theorems show that the very simple classical axiomatic presentation of the projective plane forces the existence of an underlying field of coordinates. The interested reader will find in [5], Vol. II of this *Trilogy*, a systematic study of the projective spaces over a field, in arbitrary dimension, fully using the contemporary techniques of linear algebra.

Another strikingly different approach to geometry imposed itself during the 19th century: non-Euclidean geometry. Euclid's axiomatization of the plane refers—first—to four highly natural postulates that nobody thought to contest. But it also contains the famous—but more involved—"fifth postulate", forcing the uniqueness of the parallel to a given line through a given point. Over the centuries many mathematicians made considerable efforts to prove Euclid's parallel postulate from the other axioms. One way of trying to obtain such a proof was by a *reductio ad absurdum*: assume that there are several parallels to a given line through a given point, then infer as many consequences as possible from this assumption, up to the moment when you reach a contradiction. But very unexpectedly, rather than leading to a contradiction, these efforts instead led step by step to an amazing new geometric theory. When actual models of this theory were constructed, no doubt was left: mathematically, this "non-Euclidean geometry" was as coherent as Euclidean geometry. We recall first some attempts at "proving" Euclid's fifth postulate, and then develop the main characteristics of the non-Euclidean plane: the limit parallels and some properties of triangles. Next we describe in full detail two famous models of non-Euclidean geometry: the *Beltrami–Klein disc* and the *Poincaré disc*. Another model—the famous *Poincaré half plane*—will be given full attention in [6], Vol. III of this *Trilogy*, using the techniques of Riemannian geometry.

We conclude this overview of synthetic geometry with Hilbert's famous axiomatization of the plane. Hilbert has first filled in the small gaps existing in Euclid's axiomatization: essentially, the questions related to the relative positions of points and lines (on the left, on the right, between, ...), aspects that Greek geometers considered as "being intuitive" or "evident from the picture". A consequence of Hilbert's axiomatization of the Euclidean plane is the isomorphism between that plane and the Euclidean plane $\mathbb{R}^2$: this forms the link with [5], Vol. II of this *Trilogy*. But above all, Hilbert observes that just replacing the axiom on the uniqueness of the parallel by the requirement that there exist several parallels to a given line through a same point, one obtains an axiomatization of the "non-Euclidean plane", as studied in the preceding chapter.

To conclude, we recall that there are various well-known problems, introduced early in antiquity by the Greek geometers, and which they could not solve. The most famous examples are: squaring a circle, trisecting an angle, duplicating a cube, constructing a regular polygon with n sides. It was only during the 19th century, with new developments in algebra, that these ruler and compass constructions were proved to be impossible. We give explicit proofs of these impossibility results, via field theory and the theory of polynomials. In particular we prove the transcendence of π and also the Gauss–Wantzel theorem, characterizing those regular polygons which are constructible with ruler and compass. Since the methods involved are completely outside the "synthetic" approach to geometry, to which this book is dedicated, we present these various algebraic proofs in several appendices.

Each chapter ends with a section of "problems" and another section of "exercises". Problems generally cover statements which are not treated in the book, but which nevertheless are of theoretical interest, while the exercises are designed for the reader to practice the techniques and further study the notions contained in the main text.

A selective bibliography for the topics discussed in this book is provided. Certain items, not otherwise mentioned in the book, have been included for further reading.

The author thanks the numerous collaborators who helped him, through the years, to improve the quality of his geometry courses and thus of this book. Among them, the author particularly wishes to thank *Pascal Dupont*, who also gave useful hints for drawing some of the illustrations, realized with *Mathematica* and *Tikz*.

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Chapter 1
Pre-Hellenic Antiquity

This very short chapter is intended only to give an overview of some of the first geometric ideas which arose in various civilizations before the influence of the systematic work of the Greek geometers. So *pre-Hellenic* should be understood here as "before the Greek influence".

From this "pre-Hellenic antiquity", we know of various works due to the Egyptians and the Babylonians. Indeed, these are the only pre-Hellenic civilizations which have produced written geometric documents that have survived to the present day.

It should nevertheless be mentioned here that some works in China and India—posterior to the golden age of geometry in Greece—are considered by some historians as "pre-Hellenic" in the sense of being "absent of Greek influence". But not all specialists agree on this point. Therefore we choose in this book to mention these developments at their chronological place, after the Greek period.

1.1 Prehistory

Prehistory is characterized by the absence of writing. In those days, the transmission of knowledge was essentially oral. But nowadays, we no longer hear those voices. Therefore prehistory remains as silent about geometry as it is about all other aspects of human life. The best that we can do is to rely on archaeological discoveries and try to interpret the various cave pictures and objects that have been found.

The first geometric pictures date from 25000 BC. They already indicate some mastering of the notions of symmetry and congruence of figures. Some other objects of the same period show evidence of the first arithmetical developments, such as the idea of "counting".

Particularly intriguing is the picture in Fig. 1.1: it seems to be evident that the prehistoric artist did not just want to draw a nice picture: he/she wanted to emphasize some mathematical discovery. Indeed, looking at this picture, we notice at once that:

- doubling the side of the triangle multiplies the area by 4; tripling the side of the triangle multiplies the area by 9;

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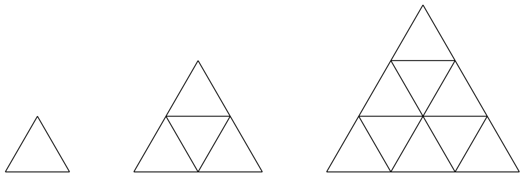

Fig. 1.1

- counting the number of small triangles on each "line" we observe that

$$1 = 1, \qquad 1+3=4, \qquad 1+3+5=9.$$

The oldest written documents that we know concerning geometry already mention the corresponding general results:

- *multiplying the lengths by n results in multiplying the areas by n^2*;
- *the sum of the n first odd numbers is equal to n^2*.

To what extent was the prehistoric artist aware of these "theorems"? We shall probably never know.

A tradition claims that the origin of arithmetic and geometry is to be found in the religious rituals of our ancestors: they were fascinated by the properties of some forms and some numbers, to which they attributed magical powers. By introducing such magical forms and numbers into their rituals they might perhaps draw the benediction of their gods.

Another tradition, reported by *Herodotus* (c. 484 BC–c. 425 BC) presents geometry as the precious daughter of the caprices of the Nile. Legendary Pharaoh *Sesostris* (around 1300 BC; but probably a compound of *Seti* and *Ramesses II*) had, claims Herodotus, distributed the Egyptian ground between "the" (by which we understand "some few privileged") inhabitants. The annual floods of the Nile valley, the origin of its fertility but also of many dramatic events, made it necessary to devise practical methods of retracing the limits of each estate after each flood. These methods were based on triangulation and probably made use of some special instances of Pythagoras' theorem for constructing right angles. For example, the fact that a triangle with sides 3, 4, 5 has a right angle seems to have been known at least since 2000 BC.

But the Nile valley certainly does not have the hegemony of early developments of mathematics, not even in Africa: the discovery in 1950 of the *Ishango* bone in Congo, dating from 22000 BC, is one of the oldest testimonies of some mathematical activity. And various discoveries in Europe, India, China, Mesopotamia, and so on, indicate that—at different levels of development—mathematical thought was present in many places in the world during antiquity.

However, up to now, modest prehistory has unveiled very little of its personal relations with geometry.

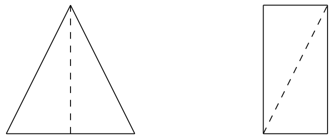

Fig. 1.2

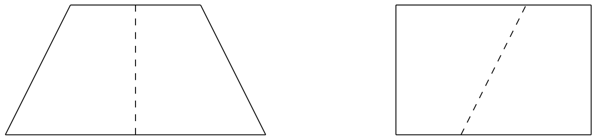

Fig. 1.3

1.2 Egypt

The oldest mathematical papyrus which has reached us is the so-called *Moscow papyrus*, most likely written around 1850 BC. But our main knowledge of Egyptian mathematics during high antiquity comes from a more extended papyrus copied by the scribe *Ahmes* around 1650 BC. These papyri contain the solutions to many arithmetical and geometrical problems whose elaboration, according to Ahmes, dates back to 2000 BC.

The *Moscow* papyrus is also called the *Golenischev Mathematical Papyrus*, after the Egyptologist *Vladimir Golenishchev* who bought the papyrus in Thebes around 1893. The papyrus later entered the collection of the *Pushkin State Museum of Fine Arts* in Moscow, where it remains today. The Ahmes papyrus is often referred to as the *Rhind papyrus*, so named after *Alexander Henry Rhind*, a Scottish antiquarian, who purchased the papyrus in 1858 in Luxor, Egypt. The papyrus was apparently found during illegal excavations on the site of the mortuary temple of Pharaoh Ramesses II. It is kept at the *British Museum in London*.

For example, Problem 51 of the Ahmes papyrus shows that

The area of an isosceles triangle is equal to the height multiplied by half of the base.

The explanation is a "cut and paste" argument as in Fig. 1.2. Cut the triangle along its height; reverse one piece, turn it upside down and glue both pieces together; you get the rectangle on the right.

An analogous argument is used in Problem 52 to show that

The area of an isosceles trapezium is equal to the height multiplied by half the sum of the bases.

See Fig. 1.3, which is again "a proof" in itself. At least, it is a "proof" in the spirit of the time: in any case, an argument based on congruences of figures.

However, let us stress that the Egyptians did not have a notion of what a "theorem" or a "formal proof" is, in the mathematical sense of the term. In particular, they did not make a clear distinction between a precise result and an approximative one. For example, one finds in Egyptian documents the following strange rule to compute the area of a quadrilateral:

The area of an arbitrary quadrilateral is obtained by multiplying the half-sums of the pairs of opposite sides.

This is in clear contradiction with Problem 52 in the Ahmes papyrus: the area of an isosceles trapezium. But this did not seem to disturb anybody. Even more amazing is the corollary (presented as such) inferred from this general "rule":

The area of a triangle is equal to the half of one side multiplied by the half sum of the other two sides.

This is again in contradiction with Problem 51 in the Ahmes papyrus, but the biggest surprise is elsewhere:

The Egyptians were able to consider a triangle as a special case of quadrilateral: a quadrilateral having one side of length zero.

This is an abstraction process which we would not have expected in those days.

Both papyri also consider the area of a circle. In Problem 50 of the Ahmes papyrus, it is claimed that

A circle whose diameter is 9 units has the same area as a square whose side is 8 units.

In view of the now well-known formula for the area of a circle, this yields the value

$$\pi = \frac{256}{81} \approx 3.16$$

... which is not that bad! As a matter of comparison, the Bible (in the first book of Kings, VII-23) gives the value 30 for the circumference of a circle of diameter 10 ... that is a value $\pi = 3$. It does not seem that the Egyptians were aware of the existence of a unique "quantity" π to be used for all circles, whatever their size. Later we shall discuss what the nature of such a "quantity" π could have been in antiquity (see Sect. 2.6).

On the other hand the Egyptians had discovered the relation (an exact relation, this time) between the length and the area of a circle.

The area of a circle is to its length as the area of the square constructed on the diameter is to its perimeter.

In modern algebraic notation, if R is the radius of the circle, this means

$$\frac{\pi R^2}{2\pi R} = \frac{(2R)^2}{4(2R)}.$$

The Egyptians also knew how to compute the volume of a pyramid:

$$\frac{1}{3} \times \text{base} \times \text{height}.$$

We do not know how they discovered this formula, but we can easily imagine how they would have made use of it.

Problem 56 of the Ahmes papyrus also investigates the "similarity" of triangles.

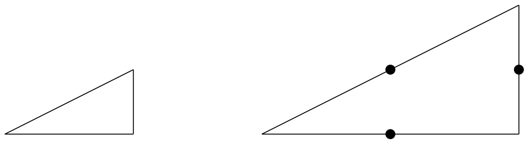

Fig. 1.4

Two right angled triangles having their respective sides proportional have their corresponding angles equal.

In this problem (see Fig. 1.4), the angles are measured by what we call today their "cotangent". Such a result was important for the construction of pyramids, in order to keep the slope constant.

1.3 Mesopotamia

Let us now leave the *Nile* valley and jump to the valleys of the *Euphrates* and the *Tigris*, in Mesopotamia. Thus we leave behind the hieroglyphs and switch to cuneiform writing (as early as 3000 BC), most often carved on clay tablets instead of papyrus: and this ensured a much better conservation through the centuries.

The Babylonians were exceptional algebraists and astronomers. They were able to solve the equations of degree one or two, and even some equations of higher degrees. We inherited from them the sexagesimal division of time and angles. Some tablets are also trigonometric tables, giving the values of the secants of the angles.

But Mesopotamia, like Egypt, did not really distinguish between exact and approximate results. A tablet (see [1]) gives the approximate areas of the first seven regular polygons. As far as the circle is concerned, it is claimed that the perimeter of the regular hexagon inscribed in a circle (= six times the radius) is equal to $\frac{24}{25}$ of the circumference. In modern notation

$$6R = \frac{24}{25} 2\pi R$$

which yields the value

$$\pi = \frac{25}{8} = 3.125.$$

Another tablet claims that the volume of a truncated cone or pyramid is obtained as the half (sic) sum of the base multiplied by the height! Egyptians had the correct formula (one third instead of one half). As far as the development of geometry is concerned we can assume there was little communication between the two civilizations.

As a matter of fact, the Babylonians were making extensive use of Pythagoras' theorem, at least one millennium before Pythagoras was born. On one tablet, one finds the following problem:

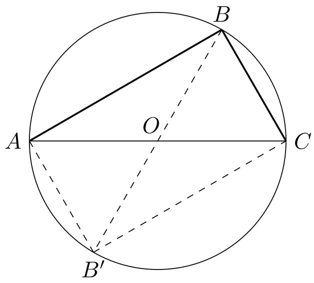

Fig. 1.5

A ladder is leant along a wall. The top of the ladder glides of three units along the wall while the foot of the ladder moves nine units on the ground. What is the length of the ladder?

Another tablet explains how to find, using "Pythagoras' theorem", the apothem of a chord inscribed in a circle.

Contrary to what was thought for a long time, the development of geometry in Mesopotamia was at least comparable, and maybe even superior, to the development of geometry in Egypt during antiquity. In particular, the systematic use of Pythagoras' theorem contrasts with its explicit absence in the Egyptian papyri.

The Babylonians also knew that

Theorem *An angle inscribed in a half circle is necessarily a right angle.*

The Egyptians were unaware of this fact, which is generally attributed to Thales, who lived a millennium later. How did the Babylonians find and justify this result? We do not know. Perhaps they made empirical observations of the following kind (see Fig. 1.5):

- Let ABC be the angle inscribed in a half circle.
- Construct B', the point symmetric to B with respect to the centre O of the circle.
- This yields four isosceles triangles, forming two equal pairs.
- Thus the four angles at A, B, C, B' are equal.
- By symmetry, the opposite sides of the quadrilateral $ABCB'$ are equal.
- The diagonals of this quadrilateral $ABCB'$ are equal as well (and equal the diameter of the circle).

This was certainly sufficient reason to convince them that the quadrilateral is a rectangle.

1.4 Problems

1.4.1 Show that a cube is the union of three equal pyramids. This yields at once the formula for the volume of one of these pyramids.

1.4.2 Consider a pyramid whose base is a square and whose summit projects orthogonally on the center of the base (like the pyramids constructed by the Egyptians). By a "cut and paste" argument, infer the formula giving the volume of such a pyramid.

1.4.3 Prove that the "ladder problem" of the Babylonians (see Sect. 1.3) has infinitely many solutions.

1.5 Exercises

1.5.1 By a "cut and paste" argument, infer the formula for the area of a parallelogram.

1.5.2 By a "cut and paste" argument, infer the formula for the area of an arbitrary triangle.

1.5.3 By a "cut and paste" argument, infer the formula for the area of an arbitrary trapezium.

Chapter 2
Some Pioneers of Greek Geometry

This chapter begins our study of the *golden pages* of the history of mathematics. These pages are the true genesis of mathematics, because—more importantly than the results that they contain—they tell us of the birth of the deductive approach in mathematics.

The famous *Elements* of *Euclid* (around 300 BC) constitute the classical reference for the study of Greek geometry. These books are a kind of "achievement" and will be devoted full attention in the next chapter. The origins of the many results organized by Euclid in a systematic and coherent theory are generally less known, or are not known at all.

In the present chapter, we focus on the efforts of some Greek geometers, anterior to Euclid, who encountered various challenging problems on their journey towards establishing the bases of geometry. In doing so, they slowly gathered all the necessary ingredients, eventually leading to the marvelous synthesis written by Euclid, but also opening the road to further work by Archimedes, Apollonius, Pappus and others. These pioneering geometers had learned from their predecessors various basic results on triangles and on the circle. What did they know exactly? Who gave for the first time a formal proof of these "empirically known" results? On which arguments and which "evidences" was such a proof based? We do not know. Nevertheless, when necessary, and to avoid repetition, we shall freely use these basic results in this chapter and refer the reader to the next chapter where Euclid's systematic treatment of these matters is presented.

Among the greatest challenges of Greek geometry, everybody has heard of the famous "circle squaring problem", and probably also of the duplication of the cube and the trisection of the angle. A final (negative) answer to these problems was found more than two millenniums later! But the many unsuccessful efforts developed to try to solve these problems resulted in the discovery of a host of techniques and results which are much more important than the problems that gave rise to them, such as, for example, the theory of conics.

Less popular—but much more fundamental for the development of geometry and eventually, of mathematics—are the *incommensurable magnitudes* and the *method of exhaustion*. In modern terms, the discovery of irrational and real numbers from

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a geometric point of view and the use of limits to handle “curved figures”. It is remarkable that Dedekind cuts had already been considered and used by the Greek geometers 23 centuries before Dedekind was born: the difference was simply that, the Greek geometers did not call them “numbers”.

2.1 Thales of Miletus

Thales of Miletus (c. 620 BC–c. 546 BC) is the first Greek geometer explicitly mentioned in the documents that have reached us. However these documents rely only on tradition, since they were written several centuries after Thales’ death. For that reason, some controversy exists concerning Thales and his work.

Thales travelled through Egypt and Mesopotamia, where he came into contact with a wealth of important scientific knowledge. This mass of information found in Thale’s brilliant mind a fertile ground upon which it was able to grow and flourish. As Thebes is given the honor of being recognized as the first Greek geometer, he has also been granted the paternity of many results that he carefully gathered during his trips. For example:

- *An angle inscribed in a half circle is a right angle.*
- *The angles at the base of an isosceles triangle are equal.*
- *When two lines intersect, the opposite angles are equal.*

Of course, these results were known long before Thales, but Thales may have been the first to have provided a formal proof of them.

Indeed, the tradition attributes to Thales the merit of having been the first one to *prove* his theorems using logical and deductive arguments. The kind of “very basic” results attributed to him, for example

- *A diameter cuts a circle in two equal pieces.*
- *Two triangles are equal as soon as a side of the first triangle is equal to a side of the second triangle, and the corresponding angles at the extremities of these two sides are mutually equal as well.*

Seem to indicate that he probably tried to infer more sophisticated results from these elementary ones. Unfortunately, no surviving document relating to Thales’ work can confirm this. Thus it is with the greatest care that we have to separate truth from various traditions and legends which depict him as a mathematician, a tradesman, a politician, an astronomer … and a champion of bachelorhood.

Aristotle (384 BC–322 BC) wrote:

> *For Thales, the fundamental question was not “What do we know?”*
> *but “How do we know it?”.*

This ideal of perfection led Greek geometers to create an unforgettable work which, for two millennia, has been considered as the highest achievement of mathematical thought.

Today, however, the name of Thales is above all attached to the following result:

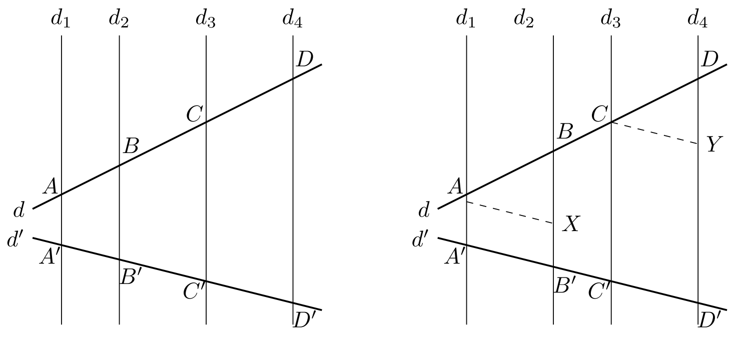

Fig. 2.1

Thales' intercept theorem *Consider two arbitrary lines d and d', and four parallel lines d_1, d_2, d_3, d_4 cutting d and d' as in the left hand diagram of Fig.* 2.1. *One has then the equality of ratios*

$$\frac{AB}{A'B'} = \frac{CD}{C'D'}.$$

Did Thales prove this result? If so, how did he prove it? We do not know. The tradition claims that Thales used this result to measure the height of the Cheops pyramid. The oldest argument which has reached us concerning this result is sometimes called the "stone age proof", since it is inspired by a picture found on a neolithic stele of around 2500 BC, but it still appears in some textbooks of the twentieth century!

- Consider first the special case depicted in the right hand diagram of Fig. 2.1, where we have chosen $AB = CD$.
- Draw AX, CY parallel to d'.
- The triangles AXB and CYD are equal, because $AB = CD$, while by parallelism, their corresponding angles are equal.
- Thus $AX = CY$ and since $AXB'A'$ and $CYD'C'$ are parallelograms, $AX = A'B'$, $CY = C'D'$, thus $A'B' = C'D'$.

This proves that

$$AB = CD \quad \Longrightarrow \quad A'B' = C'D'.$$

Now we return to the general case. Choose a unit length ε on the line d, sufficiently small to be able to measure both AB and CD with this unit. Let us say that AB has length $n\varepsilon$ and BC has length $m\varepsilon$, where n and m are two integers. By the first case, all unit lengths on d project on d' in segments of the same length, let us say, ε'. Thus $A'B'$ and $B'C'$ have respective lengths $n\varepsilon'$ and $m\varepsilon'$. This yields eventually

$$\frac{AB}{A'B'} = \frac{n\varepsilon}{n\varepsilon'} = \frac{\varepsilon}{\varepsilon'} = \frac{m\varepsilon}{m\varepsilon'} = \frac{A'B'}{C'D'}$$

and so the result is "proved".

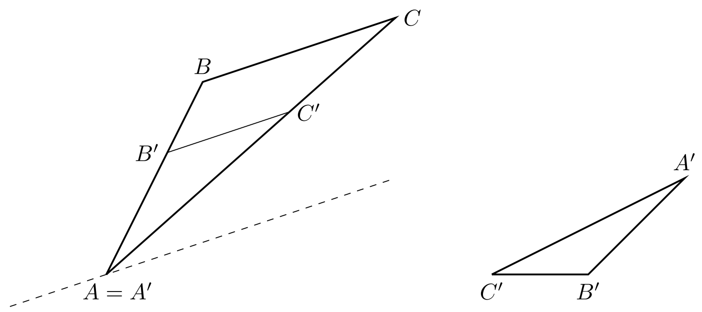

Fig. 2.2

Of course we know today that there is a big gap in this "proof": the possibility of choosing a unit ε to measure both segments AB and CD. Saying that such a unit ε exists means precisely that the ratio of the two lengths is a rational number. We shall come back to this difficulty in Fig. 2.16.

The main application of *Thales' theorem* is the theory of similar triangles (see Fig. 2.2):

Corollary *If two triangles have their corresponding angles pairwise equal, then their corresponding sides are in the same ratio.*

Indeed if the two triangles ABC and $A'B'C'$ have equal angles, respectively at A and A', B and B', C and C', "translate" the triangle $B'A'C'$ onto the triangle BAC, forcing the angles at A and A' to coincide. Since the angles at B and B' are equal as well, the lines BC and $B'C'$ are parallel and therefore Thales' theorem applies:

$$\frac{AB}{A'B'} = \frac{AC}{A'C'}.$$

An analogous argument, forcing the angles at B and B' to coincide, yields further

$$\frac{BA}{B'A'} = \frac{BC}{B'C'}$$

and so

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

This result on similar triangles played an essential role in the development of Greek geometry.