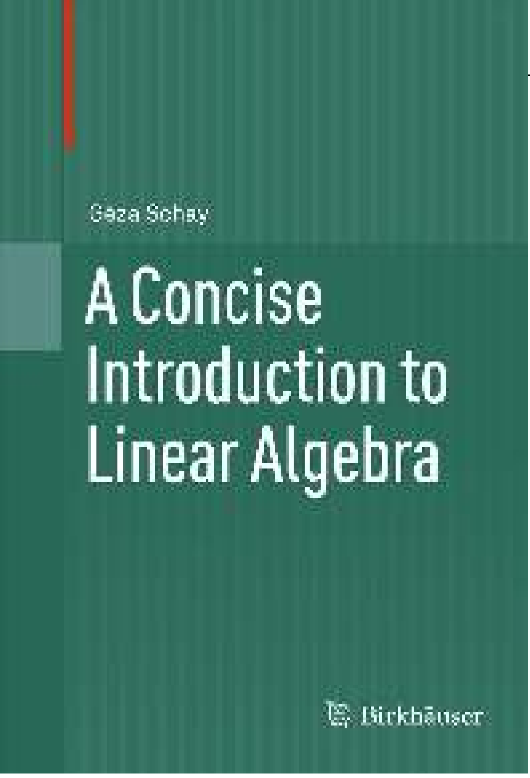
Geza Schay
A Concise Introduction to Linear Algebra
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Géza Schay
Department of Mathematics
University of Massachusetts
Boston, MA, USA

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Contents

Preface

This book is designed for a one-semester, post-calculus linear algebra course, primarily intended for mathematics, physics, and computer science majors. While basic calculus is a prerequisite for such a course, very little of it is used in the book. Certainly, multivariable calculus is not required. Vectors are treated fully in Chapter 1, but for classes familiar with them, this chapter may be skipped or just reviewed briefly. Complex numbers, series, and exponentials are presented briefly in an appendix, but they are needed only in Section 7.4, which may not be covered in some courses.

The selection of topics conforms to a large extent to the recommendations of the Linear Algebra Curriculum Study Group.[1] The main differences are that the book begins with a chapter on Euclidean vector geometry, mostly in three dimensions; determinants are treated more fully and are placed just before eigenvalues, which is where they are needed; the LU factorization is relegated to Chapter 8 on numerical methods; and the facts about linear transformations are collected in one chapter and are treated in more detail.

This book is considerably shorter than the 400 to 800 pages of most introductory linear algebra books, which are more suitable for two- or three-semester courses.

While many applications are presented, they are mostly taken from physics, and several new ones have been added in the second edition. However, these examples give only a glimpse of how the subject is used in other fields, and further details are left to texts in those fields. There is, though, a section on computer graphics and a chapter on numerical methods. Also, most sections contain MATLAB® exercises. On the other hand, we hope that the student's interest will be aroused not only by the possible applications, but also by the geometrical background and the beautiful structure of linear algebra. Nevertheless, for readers especially interested in applications, a list of the ones discussed follows this preface.

The more difficult exercises and theorems are marked by an asterisk. Some exercises are used to develop new topics, whose inclusion in the main text would have disrupted the flow of ideas. The symbols ■ and ♦ are used to indicate the end of proofs and examples, respectively.

[1] David Carlson, Charles R. Johnson, David C. Lay, A. Duane Porter. The Linear Algebra Curriculum Study Group Recommendations for the First Course in Linear Algebra. *College Mathematics Journal*, 24:1 (1993) 41–46.

In this second edition, in response to the concerns of some users of the first edition, many of the earlier proofs and explanations have been expanded and a few new ones added. Also, exercises involving laborious computations have been replaced by simpler ones, and some new ones have been added.

Foreword to Instructors

- The brevity mentioned above makes the book easier to use. Important points are not drowned in a sea of detail, and instructors and students do not have to search for what to keep and what to omit. In a minimal course, however, the following sections may be omitted entirely: Section 4.3 on computer graphics, Section 5.1 on orthogonal projections and least squares, Section 6.2 on cofactor expansions of determinants, Cramer's rule, etc., Section 6.3 on the cross product, Sections 7.3 and 7.4 on principal axes and complex matrices, and Chapter 8 on numerical methods. Theorem 3.4.8 (The Exchange Theorem) may also be omitted, since an alternative direct proof of the dimension theorem is provided in the new edition.
- The geometric content is heavily emphasized. In fact, as mentioned above, the book begins with a chapter on Euclidean vector geometry, mostly in three dimensions. Most other similar textbooks start with the solution of linear systems. We believe that this early introduction of the geometrical background helps students to visualize the concepts of linear algebra and provides easy concrete examples. Additionally, many students in this course, e.g., computer science majors, are not required to take multivariable calculus, and do not see this important material anywhere else.
- In the first chapter, the equations of planes are given in both parametric and nonparametric form, in contrast to most calculus books, which present only the nonparametric form. Many examples and exercises illustrate the transition from one form to the other. However, we avoid using the cross product at this stage, because it is only available in $\mathbb{R}^3$. We use the method of solving simultaneous equations to obtain a normal vector to a plane, and this topic is revisited as an example to Gaussian elimination. On the other hand, Section 6.3 is devoted to the cross product as an illustration of the use of determinants, and it is only at that point that it is used to obtain a normal vector to a plane.
- The "back and forth" process between parametric and nonparametric equations for lines and planes lays the groundwork for the same transition between describing a subspace of $\mathbb{R}^n$ as a set of linear combinations or as the solution set of a homogeneous system of linear equations, that is, as the column space of a matrix or the null space of another matrix. Another generalization of this issue is finding orthogonal complements of subspaces of $\mathbb{R}^n$ given in either form.
- Many books use the notation $\|\mathbf{p}\|$ for the length of a vector $\mathbf{p}$ in $\mathbb{R}^n$, but we prefer $|\mathbf{p}|$, because in $\mathbb{R}^1$ length is the absolute value, and there is no

reason to change notation for higher dimensions, just as there was none in using + for addition of both scalars and of vectors. The notation $\|\mathbf{p}\|$ is left for other norms.

- Important concepts are presented as definitions and theorems. Students are advised to memorize them. It is not enough just to understand the material; the main concepts must be remembered well to be able to build on them.
- Except for the Spectral Theorem in the complex case and theorems from other fields of mathematics, all theorems are proved. It is thus left to the instructor to adjust the level of the course from the computational to the fairly theoretical by omitting as many or as few proofs as desired.
- Great care has been taken to motivate every new concept, even those that many books do not, such as dot product, matrix operations, linear independence (not just in two or three dimensions), determinants, eigenvalues, and eigenvectors.
- The letter symbols are selected to reflect the connections between related quantities, a principle often ignored in other linear algebra books. Vectors and their components, matrices and their column and row vectors and entries are denoted by the same letters with different fonts, like $\mathbf{v}$, v_i and A, $\mathbf{a}_i$, $\mathbf{a}^j$, a_{ij}. The main exception is the unit matrix, which is, bowing to tradition, denoted by I, its columns by $\mathbf{e}_i$, and its entries by δ_{ij}.
- Only standard notation is used, so that students who study further, will have no difficulty in reading applied or more advanced texts. Nonstandard notation, such as the use of a list in parentheses for column vectors and in brackets for row vectors, or $\overrightarrow{\mathbf{a}}_i$ or $\mathbf{A}_i$ for a row vector of a matrix, found in some other introductory linear algebra books, is avoided. We use $\mathbf{a}_i$ for the column vectors of a matrix A and $\mathbf{a}^i$ for its row vectors. This is standard notation in more advanced books. (See, e.g., *Introduction to Linear and Nonlinear Programming* by David G. Luenberger, Addison-Wesley, 1973.) We also use $\mathbf{x}_A = (x_{A1}, x_{A2}, \ldots, x_{An})^T$ for the coordinate vector of a vector $\mathbf{x}$ relative to an ordered basis or basis matrix A. (Compare this, e.g., with the notation $[\mathbf{x}]_{\mathcal{B}} = (c_1, c_2, \ldots c_n)^T$ of *Linear Algebra and Its Applications* by David Lay, Addison-Wesley, 1993, where the brackets on the left are superfluous, the coordinates of $\mathbf{x}$ are denoted by the unrelated letter c, and the basis $\mathcal{B}$ is not indicated on the right, not to mention that we need an ordered basis or basis matrix here.) Our notation makes the notoriously messy topic of change of basis much simpler.
- Similarity of matrices is introduced in the context of changing bases.
- Most introductory linear algebra books introduce determinants by unmotivated formulas. This book introduces them by three simple properties, expanding on the approach in Strang.[2]

[2] Gilbert Strang, *Linear Algebra and its Applications*, 3rd ed. Harcourt Brace, San Diego, 1988.

- MATLAB exercises at the end of most sections reinforce and expand the linear algebra material. They also provide some introduction to MATLAB, but should be used in conjunction with a MATLAB manual.
- The appendix on implication and equivalence introduces the student in an informal way to certain crucial elements of proofs, and is highly recommended reading for most.
- All displayed equations are numbered, and in the new edition, mnemonic headings are appended to all definitions, theorems, figures, and examples. These numbers and headings should make references to these items easier and make their connections more transparent.

Foreword to Students

Linear algebra is probably your first mathematics course in which the theory is just as important as the computations. To study from this book you have to carefully read the text with paper and pencil in hand.

The book starts out gently, with analytic geometry, but soon the algebra takes over and the subject becomes more abstract, which may cause some difficulty for some of you.

Studying this kind of mathematics involves three interwoven steps:

1. You must understand the material.
2. You must learn the concepts thoroughly so that you remember them and can apply them knowledgeably.
3. You must practice it, doing exercises.

Each of these steps is necessary and supports the others.

In many other subjects, understanding is not a problem, and so many students believe that once they pass that hurdle, they have done enough. Not true: If you understand something in class, that does not mean you will know it the next day. You must study after every class and make sure that you are able to explain the material in your own words so that you do not forget it. If you don't, then you have to start over again on your own, with the class attendance wasted. You will need to study several hours after every class. This is especially important, because most concepts are built upon each other. For instance, vectors, introduced in Section 1.1, are used throughout the book; matrices introduced in Section 2.2 are used throughout the rest of the book, and so on.

On the other hand, you cannot do mathematics by rote memorization without understanding, because the subject is generally too complicated for that. Also, doing that would defeat the whole purpose of studying mathematics, which is the comprehension of its logic and the ability to use it in applications—not just in those that were presented, but in other similar (or even somewhat different) applications.

Working out solutions to the exercises reinforces both the learning and the understanding of the material and is often also useful in its own right, because many exercises involve important applications of the theory.

In studying linear algebra, you have to thoroughly understand and remember the definitions first, since everything else is built on them. If you don't remember a definition, you cannot possibly understand the theory that depends on it and the exercises that make use of it.

Next in importance come the theorems, lemmas (minor or auxiliary theorems), and corollaries. These are usually preceded by introductory examples and followed by further examples that illuminate various aspects and applications of the theorems. You must study these examples together with the theorems and their proofs. It is permissible to read everything just superficially at first, to get a basic understanding, but after that, you must study it again in detail. When studying a theorem, isolate the conditions or hypotheses which make it tick. Try to see where these conditions are used in the proof, and what would happen if a condition were changed or omitted. After pinpointing the conditions, do the same for the conclusions, and last, try to follow the steps of the proof. This is where the paper and pencil come in: Write these steps down. Close the book and write down the conditions, the conclusions, or the whole statement that you are studying. Try to fill in steps that are just briefly indicated in the proofs. If the proof has a reference to some earlier material, be sure to look it up and explain to yourself how it is used. The same advice applies to the follow-up examples as well: make sure you see where the conditions of the theorem are used and why they are necessary, and follow the computations on paper.

There is an appendix on implication and equivalence, which introduces in an informal way certain crucial elements of proofs. It is highly recommended reading for all those who have not seen this material before.

Finally, after you have gone through the steps listed above, you will be ready to tackle exercises. The odd-numbered ones have solutions available in a Students' Solution Manual on the book's webpage. Do those exercises first; they are usually similar to examples in the text. Don't look at the solution before making a really serious attempt to solve a problem on your own. If a problem looks too difficult at first, then look at a similar example in the text or go back and review the definition or theorem that the problem is intended to illustrate. A problem that you have solved stays much better in your mind than one that you have merely read, and its structure becomes much clearer. But, of course, once you have solved a problem, there is no harm in looking up the solution. You may even learn a different way of solving it, or find an error in your solution (or perhaps in the solution manual).

If you follow the advice above, you will probably find linear algebra to be a very interesting and enjoyable subject, but if you don't, then it may become an unpleasant chore.

Acknowledgments

I would like to express my gratitude to the following colleagues who tested early drafts of the first edition in their classes and suggested numerous, extremely valuable improvements: Guy T. Hogan, Matthew P. Gaffney, John Lutts, Seymour Kass, and Dennis Wortman. Thanks go also to Robert Seeley for a thorough reading of the first three chapters, which resulted in many helpful corrections as well. Also, I want to thank my son Peter Schay for reading the first and last drafts of the first edition and providing a long list of insightful comments.

The second edition owes many improvements to suggestions of Stephen Parrott, John Lutts, and an anonymous reviewer, which are gratefully acknowledged. Finally, I am indebted to my colleague Dennis Wortman for thoroughly checking the latest version of the manuscript and suggesting many changes, corrections, and additions of every possible type, from commas to clarifications.

Géza Schay

List of Applications

1. Center of mass. Exercises 1.1.7, 1.1.8.
2. Work as a dot product. Example 1.2.4.
3. Equations of lines and planes. Section 1.3.
4. An electrical network, Kirchhof's laws. Example 2.3.2.
5. A connection matrix for an airline. Example 2.4.8.
6. Population changes. Example 2.4.9.
7. The structure of the system expressing Kirchhof's laws. Example 3.5.7.
8. Hermite polynomials. Example 3.6.4.
9. Legendre polynomials. Exercise 3.6.11.
10. Computer graphics. Section 4.3.
11. Orthogonal projections and least-squares approximations. Section 5.1.
12. Coriolis force. Example 6.3.5.
13. Lorentz force. Example 6.3.6.
14. Systems of difference and differential equations. Section 7.2.
15. Population growth. Example 7.2.1.
16. An electric circuit with resistor, condenser, and coil. Example 7.2.2.
17. A predator-prey population model. Exercise 7.2.11.
18. Conic sections and quadric surfaces. Section 7.3.

1. *Analytic Geometry of Euclidean Spaces*

1.1 Vectors

We begin by describing some geometrical concepts. This approach may seem strange in a book on algebra, but the influence of geometry is fundamental to our subject, since the underlying geometrical ideas provide motivation, examples, and applications for the algebraic constructions. In fact, the adjective "linear" in this book's title means "pertaining to lines" (which in mathematics usually mean straight lines), and indicates the geometric origins of linear algebra.

In physics, several important notions such as displacement, velocity, and force possess not just a magnitude but a direction as well. These are typical of a large class of quantities called *vectors*, which can be depicted by arrows showing the desired directions and representing the vectors' magnitudes by their lengths. In geometry, we can use them to locate points and also, as we shall see later, to write equations of lines and planes. Let us look at a few such examples before stating formal definitions.

Example 1.1.1. (Position Vectors). Either in the plane or in three-dimensional space, consider a fixed point O and other points P, Q, R, and draw correspondingly labeled arrows $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ from O to the other points (see Figure 1.1). These arrows are called the *position vectors* or *radius vectors*

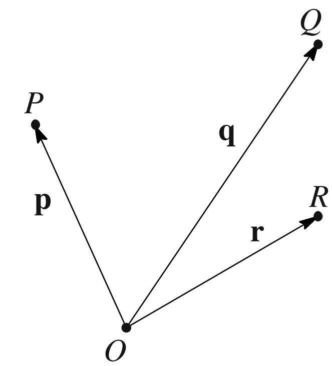

Fig. 1.1. Position vectors

G. Schay, *A Concise Introduction to Linear Algebra*,
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of P, Q, R relative to the point O, which is usually regarded as the origin of a coordinate system. Such vectors are also sometimes called bound vectors, for they are bound to the origin, in contrast to free vectors to be introduced shortly. The position vector of the point O is a special vector $\mathbf{0}$, called the zero or null vector, whose length is 0, and whose direction is undefined. ♦

Whereas in print, vectors are generally denoted by lowercase boldface letters such as $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, or by symbols like $\overrightarrow{OP}$, $\overrightarrow{OQ}$, in handwriting, boldface would be difficult and so $\underline{p}, \underline{q}$ or $\overrightarrow{p}, \overrightarrow{q}$, etc. are used instead.

Since position vectors and points are in one-to-one correspondence, you may wonder why we need position vectors at all. The answer is that various arithmetic operations that would make no sense with points can be performed with vectors, and will lend themselves to all kinds of useful constructions. Such operations are also essential for the vectors of physics.

Example 1.1.2. (Adding Forces). If, in Figure 1.2, $\mathbf{p}$ and $\mathbf{q}$ represent two *forces* acting simultaneously on a point mass at O, then the single force represented by $\mathbf{r}$, to be defined as $\mathbf{p} + \mathbf{q}$, would have the same effect. ♦

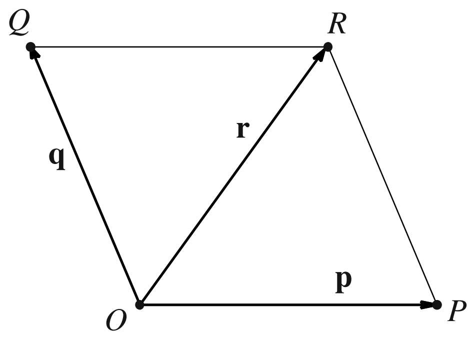

Fig. 1.2. $\mathbf{r} = \mathbf{p} + \mathbf{q}$

Example 1.1.3. (Adding Displacements). If, in Figure 1.2, $\mathbf{p}$ and $\mathbf{q}$ represent simultaneous *displacements*, then $\mathbf{r}$ represents their combined effect. This happens, for example, if a person on a boat at O walks to Q while the point O of the boat moves, together with the boat, to P (and the point Q of the boat to R). Then, as seen from the shore, the person ends up at R. ♦

The last two examples illustrate how addition of such vectors is defined. Given any pair $\mathbf{p}$ and $\mathbf{q}$ as in Figure 1.2, the corresponding points O, P, Q determine a parallelogram $OPRQ$, and the sum $\mathbf{p} + \mathbf{q}$ is defined as the diagonal vector $\mathbf{r} = \overrightarrow{OR}$. This is called the *parallelogram law* of vector addition.

A second operation we consider is multiplication of vectors by scalars. (In this context real numbers are usually called scalars, since they can be pictured on a scale, unlike vectors.) Let c be any scalar and $\mathbf{p}$ any vector, as in the previous examples. The vector $c\mathbf{p}$ is defined as the vector whose length is $|c|$ times the length of $\mathbf{p}$ and whose direction is the same as that of $\mathbf{p}$ if $c > 0$, and opposite if $c < 0$. If $c = 0$, then $c\mathbf{p}$ is the zero vector. Two examples of this type of multiplication are shown in Figure 1.3.

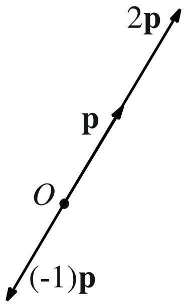

Fig. 1.3. Scalar multiples of a vector

The discussion has been somewhat informal so far, because we have not really specified very precisely the sets of vectors under consideration. It is best to remedy this omission by introducing a coordinate system into the picture.

If we consider the position vector $\mathbf{p}$ of a point P in a plane (see Figure 1.4) and introduce a Cartesian coordinate system, then we can represent the vector $\mathbf{p}$, as well as the point P, by the ordered pair (p_1, p_2) of coordinates, and write $\mathbf{p} = (p_1, p_2)$. For this representation to be of any use, we recast the parallelogram law and the multiplication of vectors by scalars in terms of the coordinates, as follows.

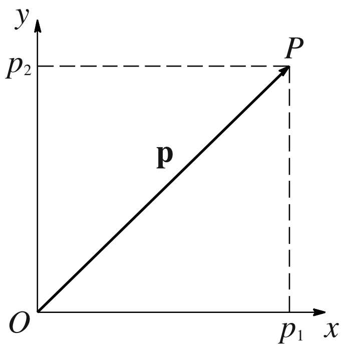

Fig. 1.4. The coordinates of a point P are the components of its position vector $\mathbf{p}$

For any two vectors $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$, Figure 1.5 illustrates that if $\mathbf{r} = \mathbf{p} + \mathbf{q}$ is the diagonal of the parallelogram spanned by $\mathbf{p}$ and $\mathbf{q}$,

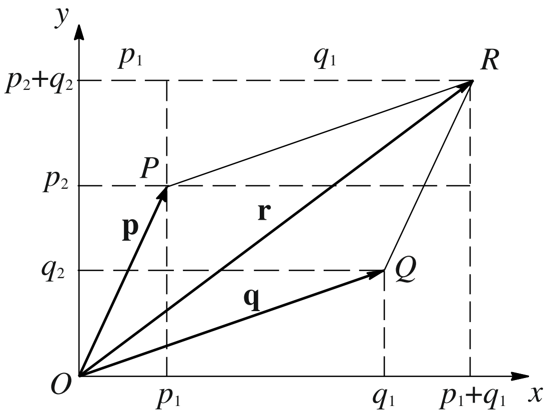

Fig. 1.5. The parallelogram law in terms of coordinates

then $\mathbf{p}+\mathbf{q}=(p_1+q_1,p_2+q_2)$ must hold; that is, we must simply add the corresponding coordinates. Similarly, we must have $c\mathbf{p}=(cp_1,cp_2)$ for every scalar c.

In light of the above discussion we now make this formal definition.

Definition 1.1.1. ***(Two-Dimensional Euclidean Vector Space).*** *The set of all ordered pairs of real numbers, together with the two algebraic operations defined below, is called the two-dimensional Euclidean vector space* $\mathbb{R}^2$.

The elements of $\mathbb{R}^2$ *are called two-dimensional vectors (or coordinate vectors) and we define the operations of vector addition and multiplication of a vector by a scalar by*

$$(p_1,p_2)+(q_1,q_2)=(p_1+q_1,p_2+q_2), \tag{1.1}$$

and

$$c(p_1,p_2)=(cp_1,cp_2) \tag{1.2}$$

for every (p_1,p_2), (q_1,q_2) *and any scalar* c.

The scalars p_1 *and* p_2 *are called the components of the vector* $\mathbf{p}=(p_1,p_2)$. *Furthermore, two vectors are said to be equal if and only if their corresponding components are equal.*

Example 1.1.4. (A Parallelogram). In Figure 1.6, let the points $P=(1,5)$ and $Q=(3,1)$ be given. Then the corresponding coordinate vectors are $\mathbf{p}=(1,5)$ and $\mathbf{q}=(3,1)$, and the position vector of the point R that makes $OQRP$ into a parallelogram is $\mathbf{r}=\mathbf{p}+\mathbf{q}=(1+3,5+1)=(4,6)$. The midpoint M of the parallelogram has the position vector $\frac{1}{2}\mathbf{r}=(2,3)$. ♦

The following simple properties follow from Definition 1.1.1 and the algebraic properties of real numbers. They will be used in the definition of a general vector space in Chapter 3.

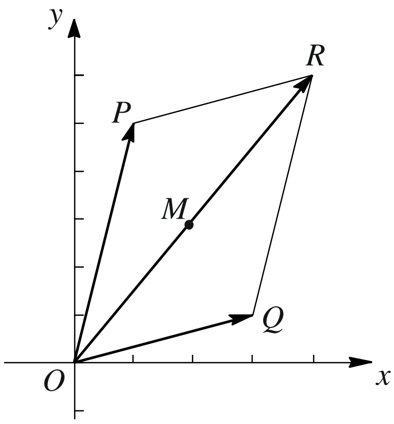

Fig. 1.6. The midpoint of a parallelogram in terms of the position vectors of the vertices

Theorem 1.1.1. ***(Basic Properties of Vectors in*** $\mathbb{R}^2$***).*** *For all vectors* $\mathbf{p}, \mathbf{q}, \mathbf{r}$ *in* $\mathbb{R}^2$ *and all scalars* a, b *we have:*
1. $\mathbf{p}+\mathbf{q}=\mathbf{q}+\mathbf{p}$ *(commutativity of addition),*
2. $(\mathbf{p}+\mathbf{q})+\mathbf{r}=\mathbf{p}+(\mathbf{q}+\mathbf{r})$ *(associativity of addition),*
3. There is a vector $\mathbf{0}$ *such that* $\mathbf{p}+\mathbf{0}=\mathbf{p}$ *for all vectors* $\mathbf{p}$ *(existence of zero vector),*
4. For each vector $\mathbf{p}$ *there is a vector* $-\mathbf{p}$ *such that* $\mathbf{p}+(-\mathbf{p})=\mathbf{0}$ *(existence of additive inverse),*
5. $1\mathbf{p}=\mathbf{p}$ *(rule of multiplication by 1),*
6. $a(b\mathbf{p})=(ab)\mathbf{p}$ *(associativity of multiplication by scalars)*[1]*,*
7. $(a+b)\mathbf{p}=a\mathbf{p}+b\mathbf{p}$ *(first distributive law),*
8. $a(\mathbf{p}+\mathbf{q})=a\mathbf{p}+a\mathbf{q}$ *(second distributive law).*

Proof. 1. $\mathbf{p}+\mathbf{q}=(p_1+q_1,p_2+q_2)=(q_1+p_1,q_2+p_2)=\mathbf{q}+\mathbf{p}$.

2. $(\mathbf{p}+\mathbf{q})+\mathbf{r}=(P_1+q_1,p_2+q_2)+(r_1,r_2)=(p_1+q_1+r_1,p_2+q_2+r_2)=(p_1,p_2)+(q_1+r_1,q_2+r_2)=\mathbf{p}+(\mathbf{q}+\mathbf{r})$.

3. Defining $\mathbf{0}=(0,0)$ we have $\mathbf{p}+\mathbf{0}=(p_1+0,p_2+0)=(p_1,p_2)=\mathbf{p}$.

4. Defining $-\mathbf{p}=(-p_1,-p_2)$ we have $\mathbf{p}+(-\mathbf{p})=(p_1+(-p_1),p_2+(-p_2))=(0,0)=\mathbf{0}$.

We leave the rest to the reader. ∎

Let us remark that Properties 2, 6, 7, and 8 can be extended to several vectors and scalars much as for numbers, and we shall use such extensions without further ado.

Subtraction of vectors can be defined just as for numbers.

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[1] "Associativity" is nonstandard here; there is no commonly used name for this property.

Definition 1.1.2. ***(Subtraction in*** $\mathbb{R}^2$***).*** *For every* $\mathbf{p} = (p_1, p_2)$, $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$, *we define*

$$\mathbf{p} - \mathbf{q} = \mathbf{p} + (-\mathbf{q}). \tag{1.3}$$

The definitions lead at once to the following alternative expressions for the negatives of vectors and for their subtraction in terms of components.

Theorem 1.1.2. ***(Negative and Subtraction in*** $\mathbb{R}^2$ ***in Terms of Components).*** *For every* $\mathbf{p} = (p_1, p_2)$, $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$,

$$-\mathbf{p} = (-1)\,(p_1, p_2) \tag{1.4}$$

and

$$\mathbf{p} - \mathbf{q} = (p_1 - q_1, p_2 - q_2). \tag{1.5}$$

Example 1.1.5. (A Subtraction in $\mathbb{R}^2$*).* Let $\mathbf{p} = (1, -3)$ and $\mathbf{q} = (-4, 5)$. Then $-\mathbf{p} = (-1)(1, -3) = (-1, 3)$, $-\mathbf{q} = (-1)(-4, 5) = (4, -5)$, and $\mathbf{p} - \mathbf{q} = (1, -3) + (4, -5) = (5, -8)$. ♦

We have the following list of further properties of vectors.

Theorem 1.1.3. ***(Properties of Vectors in*** $\mathbb{R}^2$ ***Involving 0 and Subtraction).*** *For all vectors* $\mathbf{p}, \mathbf{q}, \mathbf{x}$ *in* $\mathbb{R}^2$ *and all scalars* c *and* d *we have*
1. $0\mathbf{p} = \mathbf{0}$,
2. $c\,\mathbf{0} = \mathbf{0}$,
3. $\mathbf{p} + \mathbf{x} = \mathbf{q}$ *if and only if* $\mathbf{x} = \mathbf{q} - \mathbf{p}$,
4. If $c\mathbf{p} = \mathbf{0}$ *then either* $c = 0$ *or* $\mathbf{p} = \mathbf{0}$ *or both,*
5. $(-c)\mathbf{p} = c(-\mathbf{p}) = -(c\mathbf{p})$,
6. $c(\mathbf{p} - \mathbf{q}) = c\mathbf{p} - c\mathbf{q}$,
7. $(c - d)\mathbf{p} = c\mathbf{p} - d\mathbf{p}$.

Proof. We prove only Property 3. There are two statements here: the "if" and the "only if" part. Writing $\mathbf{p} = (p_1, p_2)$, $\mathbf{q} = (q_1, q_2)$, and $\mathbf{x} = (x_1, x_2)$, if $\mathbf{x} = \mathbf{q} - \mathbf{p}$, then we have

$$\mathbf{x} = (x_1, x_2) = (q_1 - p_1, q_2 - p_2) \tag{1.6}$$

and so

$$\mathbf{p} + \mathbf{x} = (p_1 + (q_1 - p_1), p_2 + (q_2 - p_2)) = (q_1, q_2) = \mathbf{q} \tag{1.7}$$

must hold.

Conversely, the "only if" part of Property 3 is equivalent to saying that if $\mathbf{p} + \mathbf{x} = \mathbf{q}$, then $\mathbf{x} = \mathbf{q} - \mathbf{p}$. (See Appendix 1.) So, to prove this part, assume $\mathbf{p} + \mathbf{x} = \mathbf{q}$. Then we can write this equation in components as

$$(p_1 + x_1, p_2 + x_2) = (q_1, q_2) \tag{1.8}$$

and, because the equality of two vectors means that the corresponding components must be equal, we have

$$p_1 + x_1 = q_1 \tag{1.9}$$

and

$$p_2 + x_2 = q_2. \tag{1.10}$$

Solving these equations for x_1 and x_2 and combining them into a vector, we get

$$\mathbf{x} = (x_1, x_2) = (q_1 - p_1, q_2 - p_2) = \mathbf{q} - \mathbf{p}. \tag{1.11}$$

■

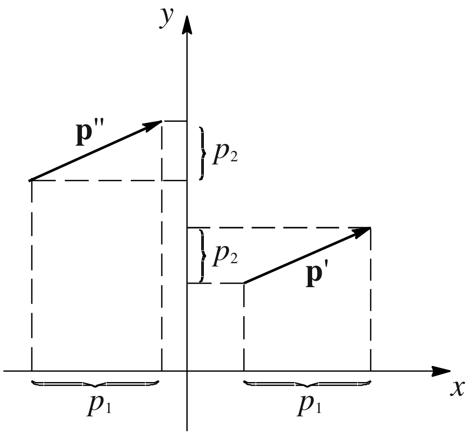

Fig. 1.7. Representative arrows of a vector in $\mathbb{R}^2$

There is an additional, important way of associating arrows with ordered pairs of coordinates. If we draw an arrow $\mathbf{p}'$ anywhere in a coordinate system (see Figure 1.7), not necessarily at the origin, then we can still project it perpendicularly onto the axes and consider the signed lengths $p_1,\ p_2$ of the projections to be the components of a vector in $\mathbb{R}^2$. Of course, any other arrow, such as $\mathbf{p}''$, obtained from $\mathbf{p}'$ by a parallel shift, will produce the same $p_1,\ p_2$ values. Thus for a given vector $(p_1, p_2) \in \mathbb{R}^2$ there corresponds a class $\mathbf{p}$ of infinitely many arrows parallel to each other and equal in length,[2] all having the same signed scalar projections p_1 and p_2. The arrows like $\mathbf{p}'$ and $\mathbf{p}''$ are

[2] Equivalence class is a standard term used for sets whose members constitute all objects equivalent to each other under a certain type of relation called an equivalence relation.

equivalent representatives of the class $\mathbf{p}$. Such classes of equivalent arrows are called *free vectors*, since the arrows can be shifted freely. We usually identify the free vector $\mathbf{p}$ with the vector $(p_1, p_2) \in \mathbb{R}^2$, that is, we write $\mathbf{p} = (p_1, p_2)$. This should not lead to confusion, just as referring to a point as (x, y) instead of a point P with coordinates (x, y) does not.

A free vector can be represented by any one of its arrows; that is, the whole class $\mathbf{p}$ is known if any member of $\mathbf{p}$ is known. Unfortunately, many people confuse the class $\mathbf{p}$ with the individual arrows, and call $\mathbf{p}'$ and $\mathbf{p}''$ equal *vectors*, rather than just equivalent representative *arrows* of the vector $(p_1, p_2) \in \mathbb{R}^2$ or of the free vector $\mathbf{p}$.

Why do we use free vectors at all? There are at least three reasons. First, they arise rather naturally as representations of coordinate vectors, as we have just seen. Second, in physical applications some vector quantities are not bound to any fixed point. (For example, the velocity vector of a non-rotating object can reasonably be drawn at any point of the object.) Third, the pictures of many constructions become simpler and less cluttered if we use well-positioned representative arrows, rather that just vectors at O. Many examples of this usage will follow, but here we just look at a variation of the addition of vectors in terms of free vectors. In Figure 1.8, let the arrows marked $\mathbf{p}$ and $\mathbf{q}$ represent the free vectors $\mathbf{p}$ and $\mathbf{q}$ (we shall abbreviate this statement from now on as customary to "let $\mathbf{p}$ and $\mathbf{q}$ be two vectors as shown"). Then, obviously, their sum is represented by the arrow $\mathbf{p} + \mathbf{q}$. This is sometimes called the *triangle law* of vector addition. If the arrows represent displacements, then it is the natural description of their sum, that is, of one displacement followed by another.

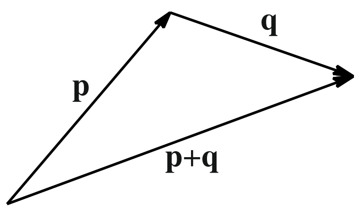

Fig. 1.8. The triangle law of vector addition

Now let us turn to the addition of several, say four, vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$ as given in Figure 1.9. By repeated application of the triangle law we get the sum as shown. (Because of the associativity of vector addition, just as with numbers, we do not need parentheses in the sum.) Contrast the simplicity of this construction with the mess we would get if all vectors were drawn at O.

Since we have $(\mathbf{p} + \mathbf{q}) - \mathbf{p} = \mathbf{q}$, we can relabel Figure 1.8 to illustrate the subtraction of vectors, by writing $\mathbf{r}$ for $\mathbf{p} + \mathbf{q}$ and $\mathbf{r} - \mathbf{p}$ for $\mathbf{q}$ as in Figure 1.10. The triangle law applied to Figure 1.10 shows that $\mathbf{p} + (\mathbf{r} - \mathbf{p}) = \mathbf{r}$, as it should be, and that $\overrightarrow{PR} = \mathbf{r} - \mathbf{p}$. This construction is especially useful for

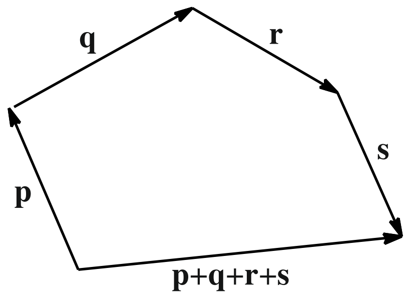

Fig. 1.9. Addition of several vectors by the triangle law

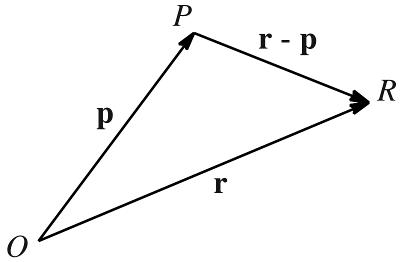

Fig. 1.10. Subtraction of vectors

obtaining the coordinate vectors of arrows joining given points as in the following example.

Example 1.1.6. (A Vector with Given Endpoints). Given two points $P = (1, 2)$ and $R = (3, 6)$ in the plane, find the coordinate vector of $\overrightarrow{PR}$.

We can write the position vectors of the given points as $\mathbf{p} = (1, 2)$ and $\mathbf{r} = (3, 6)$, and so $\overrightarrow{PR} = \mathbf{r} - \mathbf{p} = (3 - 1, 6 - 2) = (2, 4)$. ♦

Example 1.1.7. (Finding Various Points of a Parallelogram). Given three points $A = (4, 3)$, $B = (-1, 4)$, and $C = (0, -2)$ in the plane (see Figure 1.11), find the coordinates of the point D that makes $ABDC$ a parallelogram, and those of the midpoint M of the parallelogram.

The position vectors of the given points are $\mathbf{a} = (4, 3)$, $\mathbf{b} = (-1, 4)$, and $\mathbf{c} = (0, -2)$. Then, finding first $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (-5, 1)$ and $\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = (-4, -5)$, we can use them to find $\overrightarrow{AD} = (\mathbf{b} - \mathbf{a}) + (\mathbf{c} - \mathbf{a}) = (-9, -4)$. Now $\mathbf{d} = \mathbf{a} + \overrightarrow{AD} = (-5, -1)$, and this ordered pair also gives the coordinates of D. The position vector $\mathbf{m}$ of the midpoint M can be obtained as $\mathbf{m} = \mathbf{a} + \frac{1}{2}\overrightarrow{AD} = (4, 3) + \frac{1}{2}(-9, -4) = (-\frac{1}{2}, 1)$. ♦

We now define the three-dimensional Euclidean vector space $\mathbb{R}^3$ of coordinate vectors.

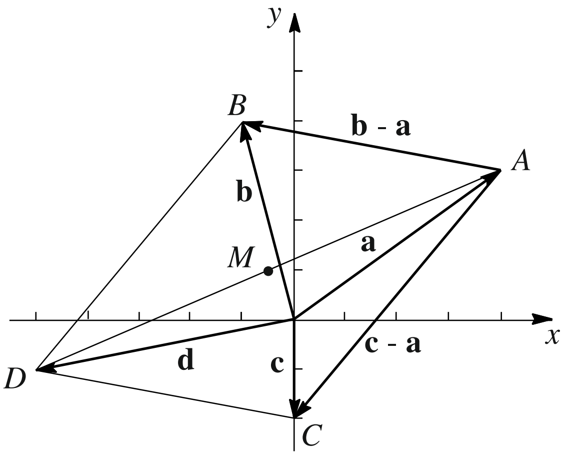

Fig. 1.11. Computing the coordinates of vertex D and midpoint M of a parallelogram, given three vertices A, B, C

Definition 1.1.3. ***(Three-Dimensional Euclidean Vector Space).*** *The vector space $\mathbb{R}^3$ is the set of all ordered triples (p_1, p_2, p_3) of real numbers with the operations defined componentwise just as in $\mathbb{R}^2$: For all (p_1, p_2, p_3), (q_1, q_2, q_3) and every scalar c,*

$$(p_1, p_2, p_3) + (q_1, q_2, q_3) = (p_1 + q_1, p_2 + q_2, p_3 + q_3), \tag{1.12}$$

and

$$c(p_1, p_2, p_3) = (cp_1, cp_2, cp_3). \tag{1.13}$$

Again, $\mathbb{R}^3$ is called the three-dimensional Euclidean vector space and its elements are called three-dimensional vectors (or coordinate vectors). The scalars p_1, p_2, p_3 are called the components of the vector $\mathbf{p} = (p_1, p_2, p_3)$ and two vectors are said to be equal if and only if their corresponding components are equal.

Just as in two dimensions, if we introduce a Cartesian coordinate system with the origin at O, then every $\mathbf{p} \in \mathbb{R}^3$ can be regarded as the position vector of the corresponding point P. (See Figure 1.12). Thus we identify the arrow $\mathbf{p}$ with the coordinate vector (p_1, p_2, p_3).

In three-dimensional space we can again represent coordinate vectors also by arrows drawn anywhere, not just at the origin, and we define free vectors much as in the plane.

About Figure 1.12, let us remark that the x-axis is meant to be interpreted as pointing out of the paper towards the reader. (This sense is not obvious: if you stare at the picture hard, you may see it as pointing into the paper.) In three dimensions, two kinds of coordinate systems are possible: the kind

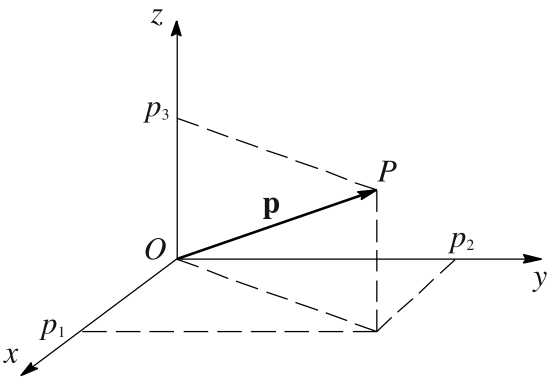

Fig. 1.12. Coordinates in $\mathbb{R}^3$

pictured here and its mirror image. The one shown is called a right-handed coordinate system, since the x, y, z axes point like the thumb, index, and middle finger of the right hand, respectively. (See Figure 1.13.) By convention, left-handed coordinate systems are rarely used.

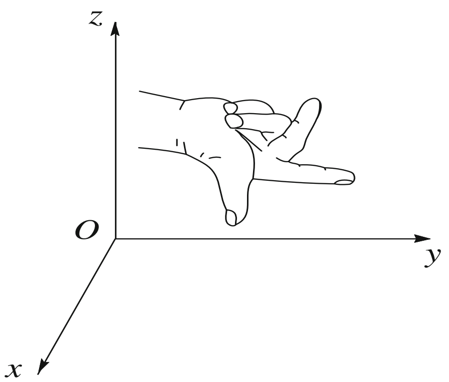

Fig. 1.13. The right hand gives the orientation of the right-handed coordinate system

Although there is no way of picturing it when $n > 3$, the n-dimensional vector space $\mathbb{R}^n$ is defined algebraically as follows.

Definition 1.1.4. ***(n-Dimensional Euclidean Vector Space).*** *For every positive integer n, $\mathbb{R}^n$ is the vector space of ordered n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers, with the basic operations defined by*

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \quad (1.14)$$